Answers

Exam IV 12/6/96

Mathematics 210 (Wilson)

Problem 1: Since you are planning ahead to accumulate the money for a purchase at the end of the payments, this amounts to a "sinking fund" as calculated using formula 9.9 on page 433 in the text. The amount to be accumulated is S = \$120,000. Payments every three months for five years make the number of payments n = 20. The annual interest rate of 8% amounts to 2% or 0.02 for each quarterly interest period, so 1 + k = 1.02. Thus the payment amount is

$$Y = \frac{.02 \times 120,000}{(1.02)^{20} - 1} = \frac{2400}{0.485947} = \$4938.81$$

Problem 2: Finding the vector $[w_1, w_2]$ of stable probabilities requires solving for numbers w_1, w_2 such that (a) $w_1 + w_2 = 1$ and (b) $[w_1, w_2] P = [w_1, w_2]$. (b) can be restated as $[w_1, w_2](P-I) = [0, 0]$. $P - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix}.$ Thus the equations w_1 and w_2 must satisfy are: From (a), $w_1 + w_2 = 1$; From (b), $-\frac{1}{2}w_1 + \frac{1}{6}w_2 = 0$ and $\frac{1}{2}w_1 - \frac{1}{6}w_2 = 0$. Writing the three equations as an augmented matrix we $\begin{bmatrix} 1 & 1 & | & 1 \\ -\frac{1}{2} & \frac{1}{6} & | & 0 \\ \frac{1}{2} & -\frac{1}{6} & | & 0 \end{bmatrix}$. Reducing the matrix produces $\begin{bmatrix} 1 & 0 & | & \frac{1}{4} \\ 0 & 1 & | & \frac{3}{4} \\ 0 & 0 & | & 0 \end{bmatrix}$ so $w_1 = \frac{1}{4}$ and $w_2 = \frac{3}{4}$.

Problem 3:

(a) From 1988 to 1995 is seven years. We need to solve $70,000(1+k)^7 = 108,000, (1+k)^7 = 108,000$ 108,000/70,000 = 1.542857143. Raising each side to the power 1/7 we get 1 + k = 1.063907018 so k = 0.063907018, approximately 6.39% annual interest.

(b) There are various ways to phrase this, but effectively you need to calculate the present value of an amount which will be \$108,000 in seven years at 5% annual interest, compounded quarterly. That can be computed as

$$\frac{108,000}{(1.0125)^{28}} = \$76,271.60$$

Problem 4: The Markov chain is absorbing, with state 1 as an absorbing state, and P is already in

canonical form. Thus $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$ and the fundamental matrix $N = (I - Q)^{-1}$ is the inverse of $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$ which is $\begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$. We are given that the chain starts in state 2, which corresponds

to the upper row of the fundamental matrix.

(a) The expected number of times the chain will be in state 2 is the entry in that upper row which goes with state 2, i.e. the first entry which is 6.

(b) The expected number of times the chain will be in state 3 is the entry in that upper row which goes with state 3, i.e. the second entry which is 4.

Problem 5:

(a) We can use formula 9.6 on page 423 to calculate the amount of the annuity. The interest rate k per pay period (one week) is 5.2%/52 = 0.1% or 0.001. The payment is Y = \$100. The number of payments is $52 \times 40 = 2080$ (weekly for forty years). Thus the amount is

$$\frac{100}{0.001}((1.001)^{2080} - 1) = \$699, 615.41$$

(b) The present value of that amount is

 $\frac{\$699,615.41}{(1.001)^{2080}} = \$87,493.99$