Problem 1: We start with the equations represented by the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

which is already in reduced form. Call the unknowns, in order, x, y, z, and w. The first, second, and fourth columns contain leading entries, but the third column does not. Since z goes with the third column, it can be assigned an arbitrary value and there are infinitely many solutions. Once a value has been chosen for z, the first row tells us that $x - z = -3$ so $x = z - 3$, and the second row tells us that $y + 2z = 4$ so $y = -2z + 4$. The last row tells us that $w = 5$. Thus the solutions are, for any choice of z, given by the four numbers $x = z - 3$, $y = -2z + 4$, z, and $w = 5$.

Problem 2: Solve the equations $x + y + z = 7$, $2x - y - 7z = 8$, $3x - 6z = 15$: The equations can be represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 2 & -1 & -7 & 8 \\ 3 & 0 & -6 & 15 \end{array}\right].$$

Adding -2 times the first row to the second row and also adding -3 times the first row to the third row gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -3 & -9 & -6 \\ 0 & -3 & -9 & -6 \end{array}\right].$$

Adding -1 times the second row to the third gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiplying the second row by $-\frac{1}{3}$ gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now adding -1 times the second row to the first gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in reduced form. Since the third column does not contain a leading entry, z can be given any value. After that value is chosen the second row tells us that $y = 2 - 3z$ and the first row tells us that $x = 5 + 2z$.

Problem 3: To find the inverse of the matrix we construct a matrix with three more columns by writing the 3×3 identity matrix next to A:

$$\left[\begin{array}{cccccc} 2 & -1 & -2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Interchanging the first and third rows gives

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -2 & 1 & 0 & 0 \end{bmatrix}.$$

Adding -3 times the first row to the second and also adding -2 times the first row to the third gives

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -3 & 0 & 1 & -3 \\ 0 & -3 & -4 & 1 & 0 & -2 \end{bmatrix}.$$

We could just multiply the second row by $-\frac{1}{2}$ to change the -2 to 1, but in order to avoid fractions we can instead subtract the third row from the second and get

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & -1 \\ 0 & -3 & -4 & 1 & 0 & -2 \end{bmatrix}.$$

Now we can add -1 times the second row to the first row and also add 3 times the second row to the third row to get

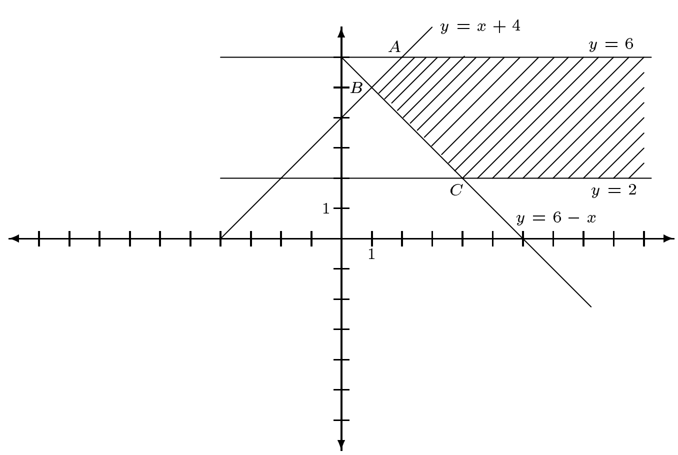
$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & -2 & 3 & -5 \end{bmatrix}.$$

Add 1 times the third row to the second, and also (afterwards) multiply the third row by -1, to get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -3 & 4 & -6 \\ 0 & 0 & 1 & 2 & -3 & 5 \end{bmatrix}.$$

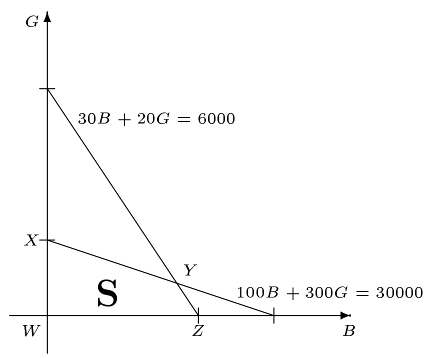
Since the left three columns have now become the 3×3 identity matrix, the right three columns must be A^{-1}.

Problem 4: We are given the inequalities $y-6 \leq 0$ (i.e. $y \leq 6$), $2-y \leq 0$ (i.e. $y \geq 2$), $6-x-y \leq 0$ (i.e. $y \geq 6-x$), and $y-x-4 \leq 0$ (i.e. $y \leq x+4$). First graph the lines $y=6$, $y=2$, $y=6-x$, and $y=x+4$, then find what side of each line agrees with the inequality. The set S is the shaded region:



The point A is on $y = 6$ and on $y = x + 4$ so $6 = x + 4$, $x = 2$, and the point is $(2, 6)$.
The point B is on $y = x+4$ and on $y = 6-x$ so $x+4 = 6-x$, $2x = 2$, and $x = 1$. Then $y = x+4 = 5$ so the point is $(1, 5)$.
The point C is on $y = 2$ and $y = 6 - x$ so $2 = 6 - x$, $x = 4$, and the point is $(4, 2)$.
Calculating $3y + x$ at A gives 20, at B gives 16, and at C gives 10. Since the region S is unbounded we cannot simply assume the minimum occurs at C, although that gave the smallest value so far. We need to check at some other point on the boundary of S on an edge which goes through C and also goes off to ∞. On the line $y = 2$ we can pick the point $(5, 2)$, one unit to the right of C, where the function gives 11 which is larger than 10. Hence the minimum value of the function is 10, taken at the point C.

Problem 5: Each stuffed Badger takes 100 square inches of fabric and 30 ounces of stuffing, while each stuffed Gopher takes 300 square inches of fabric and 20 ounces of stuffing. Thus if we make B Badgers and G Gophers we need $100B + 300G$ square inches of fabric and $30B + 20G$ ounces of stuffing. Since there are 6000 ounces of stuffing and 30,000 square inches of fabric available for each day's production, we have the constraints $100B + 300G \leq 30,000$ and $30B + 20G \leq 6000$. We can also assume it doesn't make sense to produce negative numbers of stuffed animals, so $B \geq 0$ and $G \geq 0$. (If you do not make this assumption you get the same answer but you have to deal with an unbounded feasible set.) We can then plot the feasible set S:



I have labeled the corners of the feasible set X, Y, Z, and W: Their coordinates are $X = (0, 100)$, $Y = (\frac{1200}{7}, \frac{300}{7})$, $Z = (200, 0$, and $W = (0, 0)$. Since the profit for each Badger is \$16 and for each Gopher is \$20, the profit at X is \$2000, at Y is \$3600, at Z is \$3200, and at W is \$0. Thus the maximum daily profit, \$3600, is obtained by making $\frac{1200}{7} \approx 171$ Badgers and $\frac{300}{7} \approx 43$ Gophers.