Dear Math-Ed seminarians,
Here are several selections from the writings of a French dude named Guy Brousseau, all about *epistemological obstacles*. This translation is difficult to read, so I've tried to keep it as short as possible while still including the important parts. Some parts I have marked as optional. Thanks!
-Rob

Some words to know:
A *didactical* situation is one that has been engineered for the sake of teaching something. An *adidactical* situation has no such intention to it: It occurs naturally.
Devolution sort of means explaining, but with a sense of responsibility to make sure the explained thing is actually comprehended.
The *didactical contract* between a student and a teacher outlines the responsibilities of both parties: Usually this involves the devolution of appropriate problems by the teacher and the solution of some problems by the student.

Note re the document: Rob gave me portions of a book, which I have converted to LaTeX in order to make this PDF file. I am sure I have made some errors in translation!
The book has both end-of-chapter notes and bibliographic references: I had the notes for the end of chapter 2, so I have included the pointers to those notes (as superscripts.) I did not have any expansion of the bibliographic references beyond those embedded in the text.
-Bob W.

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4. COHERENCE AND INCOHERENCE OF THE MODELLING ENVISAGED: THE PARADOXES OF THE DIDACTICAL CONTRACT

Envisaging teaching as the devolution of a learning situation from the teacher to the student has allowed us to discover certain phenomena. The attempt to model this devolution as the negotiation of a contract allows us to explain a great many of these phenomena and to predict others.

The result of these processes causes us to consider the teacher as a player faced with a system, itself built up from a pair of systems: the student and, let us say for the moment, a "milieu" that lacks any didactical intentions with regard to the student.

In the student's "game" with the milieu, knowledge is the means of understanding the ground rules and strategies and, later, the means of elaborating winning strategies and obtaining the result being sought.

In the teacher's game with the student-milieu system, the didactical contract is the means of establishing the basic rules and strategies and, later, of adapting them to changes in the student's game.

To each piece of knowledge, and perhaps to each function of a piece of knowledge, there must correspond specific situations (problems) and probably didactical contracts. The evolution of the players and of the game—unlike games having fixed rules—leads to some questioning of both knowledge and the didactical contract.

This didactique is precisely the basis of the constitution of knowledge in that the knowledge articulates the specific and the general. Before systematizing and going more deeply into this modelling, it will be useful for us to examine its coherence. This study will allow also the clarification of the functions or relationships that it would be proper to represent (by rules) and of the difficulties of the enterprise.

This paragraph will allow us to present the methodology of didactique more clearly.

Considering teaching as devolution of responsibility for the use and construction of knowledge to the student leads to some paradoxes that it is helpful to point out.

4.1. The paradox of the devolution of situations

The teacher must make sure that the student solves the problems set in order to evaluate, and make it possible to make the student evaluate, whether she has accomplished her own task.

But if the student produces her answer without having had herself to make the choices which characterize suitable knowledge and which differentiate this knowledge from insufficient knowledge, the evidence becomes misleading. This occurs particularly when the teacher is induced to tell the student how to solve the given problem or what answer to give. The student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, has not given the expected evidence of the desired acquisition. She has given only an illusion of it. The teacher has the social obligation to teach everything that is necessary about the knowledge, The student—especially when she has failed—asks her for it.

And therefore, the more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for.

This is the first paradox; it is not exactly a contradiction, but the knowledge and the teaching plan will have to proceed under a facade.

So the didactical contract faces the teacher with a true paradoxical injunction: everything that she undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion; if the teacher says what it is that she wants, she can no longer obtain it.

But the student, also, faces a paradoxical injunction: if she accepts that, according to the contract, the teacher teaches her the result, she does not establish it herself and therefore does not learn mathematics; she does not make it her own. If, on the other hand, she refuses all information from the teacher, then the didactical relationship is broken. Learning implies, for her, that she accepts the didactical relationship but that she considers it as temporary and does her best to reject it. We shall see, further on, in what way.

4.2. Paradoxes of the adaptation of situations

Let us accept that the meaning of a piece of knowledge originates to a large extent from the fact that the student acquires it by adapting to the didactical situations which are put (devolved) to her. We shall assume also that for every piece of knowledge there exists a family of situations to give it an appropriate meaning.

In certain cases, there are fundamental situations that are accessible to the student at the required time. These fundamental situations will allow her quite quickly to create a correct conception of the knowledge which can be inserted, when the time comes and without radical modification, into the construction of new knowledge.

But let us suppose that some piece o2 knowledge exists for which the above conditions are not fulfilled; there exist no situations sufficiently accessible, sufficiently efficient and in sufficiently small numbers as to allow students of any age to have access straight away, by adaptation, to a form of the knowledge that could be considered correct and definitive. It is then necessary to accept stages in the learning process. Knowledge taught by adaptation in the first stage will temporarily be not only approximate but also partly false or inadequate.

The teacher will then find herself confronted with new paradoxes such as the following.

4.2.1 Maladjustment to correctness

Even if the knowledge taught in the first stage is necessary in order for a later stage to be undertaken, the teacher must expect to find herself blamed for allowing or creating these mistakes. Reproach will come as much from her students as from teachers of higher grades, unless a tradition or cultural negotiation exonerates her.

Under the hypothesis envisaged, an alternative exists: the teacher gives up teaching by adaptation; she teaches knowledge directly in accordance with scientific requirements. But this hypothesis implies that she must give up providing a meaning to this knowledge and obtaining it as an answer to situations of adaptation because then students will colour it with false meanings.

The teacher has the choice of either teaching formal, meaningless knowledge or teaching more or less false knowledge that will have to be corrected. Intermediate choices could blend the two disadvantages and even complicate them.

The student who is being taught on the one hand a “scientific” piece of knowledge and who is being presented on the other hand with inadequate situations of reference is in a position to observe all sorts of contradictions and maladjustments between these two teaching objects. The knowledge that she obtains by understanding is even false or different from what we intend to teach her.

The distinctions that are established between theoretical knowledge and practical knowledge are perhaps often only a simple consequence and a recuperation of this purely didactical difficulty. Here, again, the student is faced with a paradoxical injunction; she must understand AND learn; but in order to learn she must to some extent give up understanding and, in order to understand, she must take the risk of not learning. Taking knowledge and its genesis (true or fictional) as a teaching object, and thus teaching knowledge and its meaning is not a perfect solution either.

4.2.2. Maladjustment to a later adaptation

Memorization of formal knowledge, largely meaningless, can be very costly in terms of learning exercises. These exercises must not reintroduce too much meaning, a fact that reinforces their difficulty. The representation which the student makes of the mathematical knowledge and its functioning is profoundly perturbed as a consequence. The more the student has been drilled in formal exercises, the more it is difficult for her, later, to restore a fruitful functioning of concepts so acquired. “Application” of learned, ready-made knowledge goes badly because the logic of the articulation of the acquisitions which compose it is exclusively that of the knowledge itself and because the role of situations has been excluded a priori.

Let us examine the alternative choice, that of a temporarily erroneous understanding of knowledge obtained by adaptation to “introductory” problems. It will be necessary to go over this knowledge again and modify it.

A new paradox appears: if the student has adapted well to the situations put to her, she has better understood the reasons for her answers and the relationship between her knowledge and the problems. It will therefore be more difficult, later on, to change this knowledge so as to make it correct and complete.

We have just shown that for some knowledge it is quite predictable that “going over” it again and modifying it will be much more difficult, because it was better learned, better understood and better consolidated during the first stage.

No doubt this fact is due to matters of a psychological nature: it is all the more difficult to change habits or opinions, since they are more intimately connected with more personal, more numerous, older activities.

But it could also be for a more directly epistemological reason. The over-adaptation of “knowledge” to the solution of a particular situation is not necessarily a factor favourable to the solution of a new situation. Too strong a differentiation, too large a dependence on direct “knowledge”, and the evolution of knowledge becomes impossible. The initial knowledge creates an obstacle. Some of these obstacles are inevitable and constitutive of knowledge—others are the result of a didactical over—investment.

Thus, under the hypothesis that some piece of knowledge is not accessible to all students by a reasonably quick adaptation to a fundamental, sufficiently correct situation, the teacher finds herself confronted by a new paradox. Whether she chooses formal teaching or teaching by adaptation, the more she insists on the learning of intermediate knowledge, the more she risks hindering further teaching, Conversely, if she abandons the establishment, the institutionalization of what is acquired, even partially, the student will find no support for the steps that follow. In some cases, the better the student adapts to an intermediate didactical situation, the greater is her maladjustment to the ensuing stage.

It is probably this phenomenon which causes teachers of higher grades to use more elementary content only in the form of procedures or algorithms and, if meaning is to be considered, to do it in situations with vocabulary and methods as different as possible from those of earlier levels.

4.3. Paradoxes of learning by adaptation

4.3.1. Negation of knowledge

Is the hypothesis that the student could construct her knowledge by a personal adaptation to an adidactical situation consistent?

Let us imagine that the teacher devolves to the student a source of autocontrollable questions or a problem. If the student solves this problem, she can think she has done so by the normal application of her earlier knowledge. The fact that she has solved the problem will seem to her to be proof that there was nothing new to learn in order to do so. Even if she is aware of having. replaced an old,

culturally-identified strategy by another of her own "invention", it will be difficult for her to claim that this "innovation" is new knowledge; what point is there in identifying it as a method since it seems it can be produced easily whenever necessary? How could a subject alone distinguish, from among all the decisions that she has made, between those which are detachable from the situation and could serve as they are in other situations, and those which are purely incidental and local?
The social conditions of learning by adaptation, by rejecting the principle of the intervention of knowledge from a third party to produce the answer, tend to make it impossible to identify this answer as new, and thus as corresponding to an acquisition of knowledge.
The subject deems trivial the question whose answer she knows insofar as she has no way of knowing whether others have considered it before, or whether no one knew how to answer it, or even whether other questions resemble it or are linked to it by the fact that they could be answered because of this one, etc. Some external person must therefore look at her activities and identify those which are interesting and have a cultural status. This institutionalization is in fact a complete transformation of the situation. To choose certain questions from among those we know how to solve, to place them at the heart of a problematique which confers the status of more or less important knowledge on the answers that these questions require and to relate them to other questions and other knowledge, ultimately constitutes the best part of scientific activity. This cultural and historical work differs totally from what it seemed necessary to leave to the student, and it comes back to the teacher. It is not, thus, the result of adaptation by the student.
In some ways, adaptation contradicts the idea of the creation of new knowledge. Conversely, knowledge is almost the cultural recognition that direct knowing is impotent to solve some situations naturally (by adaptation).

4.3.2. Destruction of its cause

Situations permitting the student's adaptation are most often repetitive by nature; the student must be able to make several attempts to investigate the situation with the help of her representations and to draw consequences from her failures or her more or less accidental successes. The uncertainty into which she is plunged is a source of both anguish and pleasure. The reduction of this uncertainty is the aim of intellectual activity and is its driving force. But knowing the solution in advance—that is to say, having transformed sufficient and particular answers into methods that give the answer every time—destroys the uncertain nature of the situation, which then loses its interest, Knowing therefore deprives the student of the pleasure of seeking and finding a "local" solution. Adaptation—by means of the knowledge—thus coincides with the renunciation of an uncertainty which in the end is pleasant. The student's adaptation tends to destroy the motivation that produces it as it tends to remove all meaning from the situation that instigates it. It should therefore end quickly and, ultimately, not occur from the moment that a process becomes necessary.
The simple image of adaptation to external disturbances is not sufficient for representing the phenomenon of learning. It leaves no room for two elements essential for maintaining the process:

- on the one hand, the creation of an intrinsic motivation which stimulates the student to search for another "occasion" for adapting herself without attempting to adapt the milieu to herself;
- on the other hand, the subject's internal adaptation without external disturbances and without real "activity" (as, for example, the resolution of the subject's internal contradictions generated by the assimilation of new schemes, as discussed by Piaget).

CHAPTER 2

EPISTEMOLOGICAL OBSTACLES, PROBLEMS, AND DIDACTICAL ENGINEERING

1. EPISTEMOLOGICAL OBSTACLES AND PROBLEMS IN MATHEMATICS

1.1. The notion of problem

1.1.1. Classical conception of the notion of problem

A student isn't really doing mathematics unless she is asking herself questions and solving problems. Everybody agrees about that. The difficulties start when questions arise about knowing what problems must be asked, who asks them, and how they are asked.
In order to simplify these difficulties, it appears that didacticians of mathematics have attempted, for some time, to project the set of imaginable problems on to a sub-space defined by the five following components.
Teacher's methodological intentions
This is the component described at the beginning of the *Livre du probleme* by Glaeser and his colleagues: problems of research, problems of training, problems of introduction, etc. (IREM de Strasbourg 1973)
Didactical intentions and objectives
Examples include Bloom's objectives: acquisition of knowledge, better understanding, analysis, and so on, (Bloom 1975).
Mathematical content
Almost always, the question consists of asking the student to establish a true formula in a theory currently being studied. The content of a problem is thus *a priori* definable as an ordered pair (T, f), T being a theory supposedly made explicit during the course, and f the formula to be found, established or located in a .; mathematical proof within T.
This conception first of all allows certain problems to be placed in relationship to others, in a partially ordered structure (a lattice), provided that we have an axiomatic system appropriate to the theory being taught; discussions about the choice of the best axiomatic system underlie most of the research that has been done on curricula for years[3]. The "best axiomatic system" would be the one which, with the least effort of learning or teaching, would allow the generation of the set of theorem-problems, of assessment or evaluation, fixed by a social consensus.
Should we provide for several specific theories, to be connected later ("classical" tendency), or for a single unifying general theory from which others are to be derived ("modern" tendency)?
Does one need many weak axioms, well organized, (Dieudonne 1964) or a few powerful axioms (Choquet 1964)? "Obvious" axioms or "highly elaborated" axioms?
In the absence of a suitable theory of knowledge accompanying a relevant learning theory, these discussions have never resulted in experimental scientific studies.
This conception furthermore allows one to distinguish, on the one hand, the ordered pair (T, f) that characterizes the problem and, on the other hand, the mathematical proof of $T \vdash f$, which can be the object of a mathematical or metamathematical study. And this distinction will serve as the basis for a new decomposition of mathematical content, following two different but closely related criteria:

- the application domain (the theory T) as opposed to the mathematical or logical "structure" operating on T;

- the mathematical model (in the sense of mathematical logic), as opposed to the language.

These pairs of opposed characteristics correspond to distinctive features which teachers use spontaneously: abstractconcrete, formal-semantic, theoretical-practical, etc., but their use has never resulted in either useful typologies or objective indices.
Mathematical component
Actually, all attempts at rational, formal description of mathematics are used to try to build intermediate variables, which, without constituting the content itself, will allow it to be generated at low cost.
The conception of problems in the form $T \vdash f$ often leads to the likening of the hypotheses to what is known, the conclusions to what is sought (or vice-versa), and the problem-solving process to a progression that will coincide easily with the mathematical proof being sought.
Some mathematical proofs can be obtained with very little effort by the application of a finite set of specifications known in advance: it is then a matter of an algorithm, an automaton. which produces the particular mathematical proof being sought.
In this case, we can construct a description, classical and beautifully simple and gratifying for the teacher, of the student's cognitive activity, of the learning and of the role of the teacher: the teacher teaches the algorithm which allows the student to establish the theorems and the student memorizes it.
Heuristic component
But for other mathematical proofs, such algorithms don't exist. In order not to lose the above acquisition model, we can imagine that mathematical proof is driven by "intuitions" which to some extent play the role of algorithms. When the implementation of an already constituted theory will provide the sought-after mathematical proof or part of it (a theorem can be applied), it is possible to rationalize these intuitions locally, the choice of theories or structures being itself guided by the heuristics which we can afterwards invoke to justify the method used. Despite their rather ad hoc character, these concepts are not without interest, as shown by other papers (Glaeser 1976, Paquette 1976, Ciosek 1976, Wilson 1976, and Janvier, 1976)[4].

1.1.2. Critique of these conceptions

The validity of such a classificatory decomposition is questionable: Regardless of the facilities that it provides, it leads to the acceptance of doubtful presuppositions ',by separating elements which function together.
The subject
The subject—the student—is absent from this conception, where she appears only as a receiver, an extremely simplified recorder in whom the acquired knowledge produces no appreciable modifications, especially not structured ones.
Sense and meaning
By the same token (and in consequence) the meaning of the mathematics disappears; that which produces not only truth, but also interest in a theorem (what Gonseth (1936) called the *idoine* character of a piece of mathematical knowledge); that which ensures that this knowledge exists as an optimal solution in the field defined by a certain setup of constraints relative to the subject and/or the knowledge itself (an object in Thom's sense: a solution to a problem (Thom 1972)); that which reveals the interest of the problem itself; etc.
The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory (semantic in Carnap's sense), not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc.
Learning

Axiomatic construction suggests an enchanted learning in which the volume of knowledge—immediately obtained, structured, useable and transferable—swells in an empty space. However ...

- A learned notion is usable only to the extent to which it is connected to others, these linkages constituting its meaning, its label, its method of activation.
- But it is *learned* only to the extent to which it is usable and effectively used, that is to say: only if it is the solution to a problem. These problems, a set of constraints to which the notion responds, constitute its meaning. It is learned only if it is "successful" and it must therefore have a territory in which it can be put into practice. This territory is only rarely general and definitive.
- Because of this localized use, the notion receives certain particularizations, limitations, deformations of language and meaning; if it succeeds well enough and long enough, it takes on a value, a consistency, a meaning, a development that make its modification, re-use, generalization and rejection more and more difficult. For later acquisitions, it becomes both an obstacle and a support.

All this demonstrates:

- why learning cannot be achieved by means of the classical scheme of continu ous and progressive acquisition (such that for any acquisition there exists a rote set of acquisitions which are equivalent to it, each of them providing a quantity of information as small as desired);

and, consequently:

- why the confusion between the algorithm for the establishment of a formula and the algorithm for the acquisition of a piece of knowledge is devoid of a basis.

Algorithm and reasoning
Several examples demonstrate all the disastrous consequences of this confusion for the learning of operations on the natural numbers.
By teaching, by the same procedures and to the same age-group, both a sophisticated theory—that of probability and statistics—and the so-called operation "mechanisms", it has been possible to show that this separation of mechanisms and reasoning is neither necessary nor useful; learning takes place by the trying out of successive, temporary and relatively good conceptions, which must be successively rejected or given an entirely new genesis each time.
If the conditions require it, the student could herself sum up complex activities as "automatisms", drawing from them meaning and possibilities of choice for her activities. But for these automatisms to be used, they must be put into place by the subject herself.
Obstacles
Those works that refer to Bachelard (1938) and Piaget (1975) also show that errors and failures do not have the simplified role that we would like them to play. Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empirist or behaviourist learning theories, but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted. Errors of this type are not erratic and unexpected, they constitute obstacles, As much in the teacher's functioning as in that of the student, the error is a component of the meaning of the acquired piece of knowledge.

1.1.3. Importance of the notion of obstacle in teaching by means of problems

Interactions
We assume, then, that the construction of meaning, as we understand it, implies a constant interaction between the student and problem-situations, a dialectical interaction (because the subject anticipates and directs her actions) in which she engages her previous knowings, submits them to revision, modifies them, completes them or rejects them to form new conceptions. The main object of didactique is precisely to study the conditions that the situations or the problems put to the student must fulfil in order to foster the appearance, the working and the rejection of these successive conceptions.
We can deduce from this discontinuous means of acquisition that the informational character of these situations must itself also change in jumps.
Conditions
Under these conditions, the didactical interest of a problem will depend in an essential way on what the student will engage in, what she will put to the test, what she will invest. It will depend on the importance for her of the rejections that she will be led to make, and of the forseeable consequences of those rejections, and on how often she would risk committing these rejected errors and on their importance.
Thus, the most interesting problems will be those which will permit the overcoming of a real obstacle. This is why, in connection with problems, I wanted to examine the question of obstacles in didactique.

1.2. The notion of obstacle

1.2.1. Epistemological obstacle

The mechanism of acquisition of knowledge, as we described it above, could be applied just as well to epistemology or to the history of science as to learning or teaching. In both cases, the notion of obstacle appears fundamental to the consideration of the problem of scientific knowledge.
According to Bachelard (1938), who was the initiator of this idea:
"It is not a question of considering external obstacles like the complexity or the transient nature of phenomena, nor of implicating weakness of the human senses and the human mind; it is in the very act of intimately knowing that there appear by a sort of functional necessity sluggishness and troubles... we know against a previous knowing" (ibid. p. 13).
Bachelard studies obstacles in physics and identifies the following: the obstacle of first experience; the obstacle of general knowledge; verbal obstacles; the obstacle of improper use of familiar images; the obstacle of unitary and pragmatic knowledge; the substantialist, realist, animist obstacle; the obstacle of quantitative knowledge.
These obstacles have survived for a long time. They probably have their equivalent in the child's thinking, even though the current material and cultural environment has without doubt somewhat modified the conditions in which they are met[5].
In mathematics, very important epistemological work has been undertaken by Althusser (1967), Badiou (1972), Houzel et al. (1976), etc., in conditions similar to those of Bachelard.
This has not yet led to a list similar to Bachelard's, but important features are becoming clearer, as well as classes of obstacles. The notion of obstacle itself is being formed and diversified. It is not easy to put forward pertinent generality on this subject, it is better to do these studies case by case. Alongside the work of recording and describing significant obstacles to the constitution of concepts, studies are developing that bear on the characteristics of the functioning of knowledge, simultaneously as support and as obstacle (alternatively or dialectically).
In addition, the notion of obstacle tends to extend outside the strict field of epistemology: in didactique, in psychology, in psychophysiology, etc.

1.2.2. Manifestation of obstacles in didactique of mathematics

Errors

An obstacle is thus made apparent by errors, but these errors are not due to chance. Fleeting, erratic, they are reproduceable, persistent. Also, errors made by the same subject are interconnected by a common source: a way of knowing, a characteristic conception, coherent if not correct, an ancient "knowing" that has been successful throughout an action-domain. These errors are not necessarily explainable.

What happens is that they do not completely disappear all at once; they resist, they persist, then they reappear, and manifest themselves long after the subject has rejected the defective model from her conscious cognitive system.

Example: A student uses the following "theorem": "If the general term of a series tends towards zero, the series converges". Is she distracted? Is she reciting incorrectly-interchanging hypothesis and conclusion-a theorem from the course? Has she misunderstood the notion of limit? Or that of series? Is it an error about necessary and sufficient conditions?

In looking at this error along with some others, we understand that in an unconscious manner this student carried out a certain reasoning, distorted by an incorrect representation of real numbers that goes back to teaching at the primary and middle school levels. The reasoning is something like the following: "If x_i tends towards zero, there exists a term n beyond which the x, are negligible. Beyond this n, practically nothing more is added, and therefore the series converges". Maybe this student would not write this reasoning down without realizing that it is wrong, but for all that, it does seem obvious to her, because she depends on certain practices which were constant in the primary and secondary teaching; only "reasonably long" numbers are written explicitly, that is to say, decimal numbers: $d = \sum_{i=n}^{m} a_i \times 10^i$, such that m and $n < 10$ (more often than not $n \in [2, 3]$).

Other numbers are designated by letters or represented—for practical reasons—by a nearby decimal which is presented as *the* approximate decimal or as even *the* number.

Example: $\pi = 3.14$.

If questions of incommensurability are raised, they are explained as being provocative or paradoxical and, in the end, gratuitous: example, "Does 1 =0.9999...?" And among the proofs put forward (generally reasoning by recurrence), the students only accept those in which the number of decimal places stays reasonably small[6].

Everything reinforces the idea that we use only a *discrete* set of numbers, and the false idea that there exists an $n \in \mathbb{N}$, such that $\forall x \in \mathbb{R}$, there exists $d \in \mathbb{D}$ such that $|x - d| < 1/10^n \Longrightarrow x = d$, (that is: x is "practically replaceable" by d, $x - d$ is zero...).

Does this idea result from a "wrong" definition of decimals carried since the elementary school? We shall come back to this question later on.

Overcoming an obstacle

The obstacle is of the same nature as knowledge, with objects, relationships, methods of understanding, predictions, with evidence, forgotten consequences, unexpected ramifications, etc. It will resist being rejected and, as it must, it will try to adapt itself locally, to modify itself at the least cost, to optimize itself in a reduced field, following a well known process of accomodation.

This is why there must be a sufficient flow of new situations which it cannot assimilate, which will destabilize it, make it ineffective, useless, wrong; which necessitate reconsidering it or rejecting it, forgetting, it, cutting it up—up until its final manifestation.

Furthermore, the overcoming of an obstacle demands work of the same kind as applying knowledge, that is to say: repeated interaction, dialectics between the student and the object of her knowledge. This remark is fundamental to the determination of what is a true problem: it is a situation that allows and motivates this dialectic.

Informational characteristics of an obstacle

A piece of knowledge, like an obstacle, is always the fruit of an interaction between the student and her surroundings and more precisely between the student and a situation which makes this knowing "of interest". In particular, it stays "optimal" in a certain domain defined by the numerical "informational" characteristics of the situation.
For example, the solution of linear systems by substitution, which is useful when there are only two equations, becomes materially impractical for a sufficiently large number of them.
Knowledge, people and their milieu being what they are, it is inevitable that these interactions lead to conceptions which are "erroneous" (or correct locally but not generally). However, these conceptions are controlled by conditions of the interaction that can be more or less modified. It is the object of didactique to understand these conditions and to use them.
This observation has important consequences, especially for teaching: if we wish to destabilize an embedded notion, it is advantageous for the student to invest her conceptions sufficiently in situations

- that are numerous and important for her,
- and above all, with sufficiently different informational conditions for a qualitative jump to be necessary.

Example: A six-year-old child knows how to distinguish numbers up to 4 or 5 with the help of procedures based on perception. These procedures very quickly become costly and less reliable as soon as the number of objects reaches 6 or 7. They fail beyond this number. If we try to teach the numbers 6, 7 and 8 in this order, we come up against numerous and increasing difficulties and a period of disarray appears.
On the other hand, if we propose the comparison of sets of from 10 to 15 objects, the perceptive model is so obviously disadvantageous that the child rejects it straight away and comes up with new strategies (term by term correspondence). What we could call intuition is often unconscious understanding of the informational limitations of knowledge modes.

1.2.3. Origin of various didactical obstacles

Origin of an obstacle
We shall now consider obstacles which come up in the didactical system. These obstacles to the student's appropriation of certain notions could be due to several causes. It is difficult to put the blame on only one of the systems of interaction. That is another consequence of the conception of learning described above. Thus, the notion of epistemological obstacle tends, in certain cases, to take the place of that of error of teaching, or of insufficiency of the subject, or of intrinsic difficulty of the knowledge,
However, we can try to distinguish various origins by looking into the subsystem (of the teacher–student–knowledge system), where its modification could overcome the obstacle, even when no modification of other systems would allow it to be avoided.
We shall thus find didactical obstacles:

- - of ontogenic origin,
- - of didactical origin and
- - of epistemological origin.

For the above example (concerning the acquisition of the notion of number) we shall talk rather of a neurophysiological limitation than of an obstacle.
Obstacles of ontogenic origin

Obstacles with an ontogenic origin are those which arise because of the student's limitations (neurophysiological ones among others) at the time of her development. She develops knowings appropriate to her abilities and goals at a particular age. Genetic epistemology provides evidence of stages and means of development (accomodation and assimilation), which at the same time resemble the stages of development of concepts in the rules of regulations which cause them to appear, and differ from them in the exact nature of the limitations which determine these regulations.

Obstacles of didactical origin

Obstacles of didactical origin are those which seem to depend only on a choice or a project within an educational system.

For example, the current presentation of decimals at the elementary level[7] is the result of a long evolution within the frame of a didactical choice first made by the encyclopedists and then by the *Convention*[8] (following a conception which goes back to Stevin himself). Because of their utility, decimal numbers were to be taught to everyone as soon as possible, associated with a system of measurement, and related to technical operations with whole numbers. As a result, for students today decimal numbers are "whole numbers with a change of units" and therefore "natural" numbers (with a decimal point) and measures. And this conception, supported by a mechanization by the student, will, right up to university level, be an obstacle to the proper understanding of real numbers, as we mentioned previously[9].

It is characteristic that the principal factor in discriminating among students in a recent questionnaire developed by *IREM de Rouen* is calculation involving decimal numbers and multiplication by a power of ten at the same time. Thus, it is the "understanding" of the very definition of decimal numbers which explains students' behaviour. Nowadays, such obstacles have become both didactical and sociocultural.

Obstacles of epistemological origin

Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative role in the knowledge being sought, They can be found in the history of the concepts themselves, This doesn't mean that we must amplify their effect or reproduce in the school context the historical conditions under which they were vanquished.

1.2.4. Consequences for the organization of problem-situations

The conception of learning which relies on the study of the development of knowledge in terms of obstacles differs appreciably from the classical conception, especially concerning the role and organization of problem-situations. And this, even more than the problem, will play a fundamental role in the process.

Motivations—conditions

The posing of a problem consists of finding a situation with which the student will undertake a sequence of exchanges concerning a question which creates an "obstacle" for her, and from which she will derive support for her acquisition or construction of a new piece of knowledge.

The conditions under which this sequence of exchanges is displayed are initially chosen by the teacher, but the process must quickly move into the partial control of the subject, who will, in her turn, "question" the situation. Motivation is generated by this investment and maintains itself by it. Instead of being a simple external motor, in balancing frustration, it builds up both the subject (her word) and her knowledge,

Thus the resolution of a problem will be for the student a kind of experimental path, the opportunity given to "Nature" (here to mathematical concepts) to manifest itself during the student's activities.

Dialectical character of the process of overcoming an obstacle

The process of overcoming an obstacle necessarily includes a series of interactions between the student and her milieu. This series of interactions makes sense only to the extent to which they relate to *the same project* (for the student) *with respect to a concept* in the genesis of which they form a stage and on which their meaning is based.

These interactions bring systems of representation into play for the student, and they can often be interpreted as exchanges of messages, even with something as apparently "amorphous" as a problem, because the student is capable of anticipation and can give direction to her actions. Consequently these interactions take on a dialogic character (a fortiori when the teacher is involved in it). Moreover, this "exchanged" information is received as facts confirming or denying hypotheses, or even as assertions. If one assumes that a piece of knowledge establishes itself by opposing another one while relying on and replacing it, one will understand that it could be said that the process of overcoming an obstacle has a dialectical character: dialectics of *a priori* and *a posteriori*, of knowledge and action, of self and others, etc.

Organizing the overcoming of an obstacle will consist of offering a situation which is likely to evolve and to make the student evolve according to a suitable dialectic. It will be a question not of communicating a piece information that we wish to teach but of finding a situation in which it is the only satisfactory or optimal one—among those with which it is competing—for obtaining a result in which the student is invested.

That is not sufficient: the situation must immediately allow the construction of an initial solution or of an attempt in which the student invests her current knowledge. If the atttempt fails or is inappropriate, the situation must nonetheless produce a new situation, modified by this failure in an intelligible but intrinsic way; that is to say, not depending arbitrarily on the aims of the teacher. The situation must allow the voluntary repetition of the testing of all the student's resources. It must be selfmotivated by a subtle game of intrinsic sanctions (and not by extrinsic sanctions that the teacher links to the student's progress). The unwinding of the learning process cannot therefore be programmed—only the situation and its choice can.

For the didactician, the question in hand is the one of simultaneously identifying a stage of a concept and a situation which poses a question for the student (one of the student's own) to which this stage is a "constructible" answer in the student's system.

In the student's functioning, we have been led to distinguish three types of questions which call for three types of didactical situation.

Different types of problem: validation, formulation, action ...

a) Questions of validation,

The student must establish the validity of an assertion. She must address herself as one subject would address another one capable of adapting or rejecting her assertions, asking her to advance proofs of what she is suggesting, or challenging her by advancing other assertions. These exchanges help to make mathematical theories explicit, but also help to establish mathematics as a means of testing them as they are formed.

A proving process is constructed in a validation dialectic that leads the student, successively, to use rhetorical figures spontaneously and then to reject them. The relationships that the student must be able to establish for this are specific to this dialectic (Brousseau 1970).

A validation problem is much more a problem of comparison of evaluations, of rejection of proofs, than it is of searching for a mathematical proof.

b) Questions of formulation.

For its validation process, thought must be based on preliminary formulations, even if they will have to be modified later. Languages, also, develop in less specific dialectics than those of validation. Communication (and its constraints) play a large role in this, one which is partly independent of problems of validation, at least of explicit validation, because, to bring about the relevance of language, this communication must be subjugated to the fulfilment of a role which submits it to pragmatic validations. It is within this framework that the economical constraints that determine judicious mathematical choices are most visible,'

c) Questions of action.

Questions of action or of mathematical decisions are those—where the sole criterion is the appropriateness of the decision—the elaboration system of this decision as well as its justification can

remain totally implicit. There is no constraint here either of formulation or of validation. It is the most general dialectic; others are only particular cases. It ends up with the subject constructing regularities, schemas, models of action—more often than not unconsciously or implicitly.
Dialectics and obstacles
Certainly, none of these dialectics is independent of the others—on the contrary.
Formulation is often facilitated if an implicit model of action exists: the subject knows better how to formulate a problem if she has been able to solve it.
Action is facilitated by a suitable formulation, as Vygotsky (1983) has shown. Language cuts the situation up into relevant objects and relationships. Action provides one fundamental type of implicit validation, and formulation another.
But, conversely, each domain can be an obstacle to progress within the other domains. Some things are better done than said. Implicit models are better able to take a larger number of facts at the same time, and are more versatile and easier to restructure. Conditions that are too favourable to action make explanation useless. For example, when the Babylonians' hexagesimal systems for astronomical calcu lations was in use, the need for the decimal point did not arise, nor for the name of the reference unit, for an error of 1 to 60 was unthinkable to people who knew what they were talking about. In the same way, a language that is "too easy" to handle
(End of paragraph missing... RLW)

2. EPISTEMOLOGICAL OBSTACLES AND *DIDACTIQUE* OF MATHEMATICS

2.1. Why is *didactique* of mathematics interested in epistemological obstacles?

The transposition into mathematics of the notion of epistemological obstacle, which Bachelard (1938) thought of confining to the experimental sciences, was made possible and even necessary by the development of the Theory of Didactical Situations in the 1970s, It came directly from the concept of the "informational leap" (Brousseau 1974a) and from the "theorems" of *didactique* which flow from it.
A piece of knowledge is the result of the student's adaptation to a situation, S, which "justifies" this piece of knowledge by making it more or less effective, of different pieces of knowledge leading to learning and of the performance of tasks of different complexity. Depending on the values of variables relevant to S, one can envisage the association of each useful piece of knowledge in S with a region of effectiveness (or cost). The upper envelope of this region can include maxima, separated by saddle points (or any other singularity). Thus, in order to make the student create a particular piece of knowledge, the teacher "must" choose values which make this piece of knowledge optimal with respect to competing pieces of knowledge; progression is by leaps and not smooth. For example, if one wishes to encourage the solution of a linear system by means of linear combinations, it is better to choose systems of rank 4 rather than 2 or even 3 for students who know the method of substitution.
This reasoning can be applied to the analysis of the historical genesis of a piece of knowledge as well as to the teaching of it or to a student's spontaneous evolution.
Learning by adaptation to the *milieu* must therefore bring about cognitive ruptures: adaptation and changes in implicit models, in languages, and in cognitive systems.
If her history binds a student—or a cultural group—to a step-by-step progression towards a pass, the principle of adaptation can itself counter the rejection, however necessary, of an inadequate piece of knowledge. This fact suggests the idea that "transitory" conceptions endure and persist.
In a procedure opened up by Gonseth (1936), these ruptures can be predicted by the direct study of situations (the effects of didactical variables) and pieces of knowledge and not only by the (indirect) study of students' behaviour (Brousseau, 1974a, 1976).

According to Salin (1976), however, taking this route requires the re-examination of the interpretation of students' errors and the ways in which they are produced. Until this suggestion was put forward, such errors were all attributed either to erratic disfunctioning or to the absence of knowledge, and therefore viewed very negatively. Recurrent errors should now be envisaged as the result of (produced by and constructed around) conceptions, which, even when they are false, are not accidental but are often positive acquisitions.
From the outset, therefore, researchers should

a) find recurrent errors, and show that they are grouped around conceptions;

b) find obstacles in the history of mathematics;

c) compare historical obstacles with obstacles to learning and establish their epistemological character.

2.2. Do epistemological obstacles exist in mathematics?

On the first point (finding recurrent errors), observations of striking errors have been developed: [$(a+b)^2 = a^2 + b^2$; $0 \cdot a = a$; $\sqrt{a^2} = a$; $(0.2)^2 = 0.4$; and so on], but their linkage with conceptions depends on statistical methods which necessitate adjustment to standard methods (Cronbach, 1967; Pluvinage, 1977; Gras, 1979). Progress has been made possible by a better definition of the notion of conception based on the Theory of Didactical Situations.
The possibility of provoking the acquisition of different conceptions is demonstrated for rational numbers (Brousseau 1980, 1981; Brousseau and Brousseau, 1987): either *measurement* or *subdivision* is obtained by simple manipulation of didactical variables. Ratsimba-Rajohn (1981) observes how these two conceptions can become mutual obstacles and even coexist for the same student and how an initial conception can be reinforced, rather than rejected, in spite of an *a priori* sufficient informational leap.
On the second point (finding obstacles in the history of mathematics), Glaeser's study (1981) on the history of signed numbers unquestionably shows the interest and the importance of these phenomena of rupture—observable in the history of mathematics—for the understanding of students' difficulties. But it then becomes ;,apparent that Bachelard's model (1938) has to be interpreted before it can be extended to mathematics. Duroux (1982) proposes not a definition, but a list of necessary conditions:

a) An obstacle is a piece of knowledge or a conception, not a difficulty or a lack of knowledge.

b) This piece of knowledge produces responses which are appropriate within a particular, frequently experienced, context.

c) But it generates false responses outside this context. A correct, universal response requires a notably different point of view.

d) Finally, this piece of knowledge withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better piece of knowledge is not sufficient for the preceding one to disappear (this distinguishes between the overcoming of obstacles and Piaget's adaptation). It is therefore essential to identify it and to incorporate its rejection into the new piece of knowledge.

e) After its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way.

On the third point (comparison of historical obstacles with learning obstacles), the results are starting to appear substantial. On the notion of limit, let us cite the very fine remarks of Berthelot and Berthelot (1983) and Sierpinska's (1985, 1987) important observations; and on the simple continuity of functions, let us mention El Bouazzaoui's (1988) second and recent thesis on the conceptions of teachers, students, and manuals and those which appear in the history of mathematics. This work leaves little doubt: obstacles certainly exist, even if distinguishing them, recognizing them, listing them, and examining their relationships and their causes requires much more discussion and research.

Fundamentally cognitive obstacles seem able to be ontogenetic, epistemological, didactical and even cultural, according to their origin and the way in which they evolve. Perhaps it would be interesting also to differentiate among them according to the form of control of the knowledge (protomathematical, paramathematical, or mathematical) where the rupture is produced.

2.6. Obstacles and fundamental didactics

But obstacles also pose more fundamental didactical problems. In fact, if the student's acquisition of mathematical knowledge *necessarily* occurs according to the scheme of a succession of different conceptions, each more or less forming an obstacle to the following one, then many didactical practices justified by the simply additive classical model must be reviewed and perhaps rejected. But this model affects both internal (within and between classes) and external (between teachers and the society) negotiations with the educational system, in terms of the teaching curriculum.

2.6.1. Problems internal to the class

The function of a lesson is no longer only to introduce a new piece of knowledge that it is necessary to learn, which is placed harmoniously next to preceding knowledge, but it is also to encourage the forgetting—or even explicitly the destructionof old conceptions which had their use but which have become incompatible with this new knowledge.

It is not only a technical problem; the didactical problem is completely different. Not only the diagnosis of errors, their explanation, and the description which follows are modified, but also the reassignment of duties and responsibilities between the teacher and the students.

In order to indicate the size and difficulty of the problem, let us examine, only as examples, two very important aspects of the didactical contract that are open to complete transformation.

a) *How can the teacher accept that the result of her teaching is knowledge* on the student's part, which she knows is not only incomplete but also false and will be refuted by what follows. For that acceptance, serious support will be needed, not only from the institution, but also from the culture and from the society. How can the student herself have confidence in a contract which sooner or later will contain such clauses?

b) *The importance of memory of the circumstances of the teaching* cannot be ignored and can no longer be excluded from the teachers' responsibility as it is today.

This point merits an explanation. Epistemological obstacles do not reside in the formulation of institutionalized knowledge (teaching tends to communicate a "clean" knowledge in this respect) but in the representations which the subjectand eventually the teacher—construct in order to ensure the functioning and the understanding of the knowledge. This understanding is linked to the circumstances of the learning and is necessary for the implementation of institutionalized knowledge. The student must therefore remember: not only the knowledge which is taught to her but also the circumstances of the learning, this memory being organized in her own way. This memory is at present solely the student's responsibility. The responsibility of the educational system on this subject is limited to the organization of institutionalized knowledge and an *ad*

hoc progression. It allows the regulation of questions of temporal dependence in such a way that it is possible for some teachers to produce sequences of worksheets, lessons or exercises without saying anything about their inter-relationships. One could even envisage a succession of different teachers, each doing her hour of teaching knowing nothing at all about the student's past except her institutionalized knowledge. From that point, recognition et the existence of epistemological obstacles and the wish to handle them "officially" in the didactical relationship brings the teacher to recognize her student's history, the historicity of her knowledge and her standing. This recognition causes her, as a participant in and witness of this history, to summon up a different didactical memory. She must remember the context, the examples and the behaviours, and especially the meaning that she causes to develop. Part of this knowledge is institutionalized, at least at the level of the class, and there exists a small inventory of common images held and recognized by the culture. Recollection of some of the students' personal behaviours is essential. This recognition of an item of knowledge in evolution, and of all that this implies, is clearly visible in the experiment on rational numbers cited previously. It is clear that this still personalized and contextualized knowledge cannot be mobilized by students without the support of a witness who has a recollection of the conditions of previous learning.
These *difficulties* are even more visible and acute when they concern *relationships between levels.* Only reference to culturally recognized examples can replace a memory of the context of the previous class; when this reference is impossible, a whole section of the student's acquisition is lost. Ignoring earlier learning is also a way of avoiding the discussion which the revival of old knowledge provokes and therefore of ignoring epistemological obstacles.

2.6.2. Problems external to the class

Integration of this new model of didactical communication requires modification of the teachers' epistemology. But this serves as a basis for negotiation between teachers and students, and also with the noosphere and the whole public. One can imagine the magnitude of the social and cultural modifications which would be produced by changes in this domain.
The fact that these items of knowledge, even false ones, can be necessary to support the establishment of definitive knowledge is difficult to accept and negotiate. The idea of over-learning, of lateness of initiation, or of loss accrued by change of class, throws custom into disorder. How can one obtain such a contract without straining even further the difficult, legitimate and necessary control that society has on the communication of knowledge?
NOTES

1. Perrin-Glorian quotes Brousseau.

2. Glaeser reacted to Brousseau's introduction of the concept of "epistemological obstacle" in *didactique* in a discussion about the epistemology of negative numbers (Glaeser, 1981). He objected that such a theorization was premature considering the state of our knowledge at the time. The issue raised by Glaeser and made explicit in his reaction to (Brousseau 1976) is whether it is legitimate or not to consider epistemological obstacles as genuine knowings ("*connaissances*", Glaeser 1984).

3. *Editor's note:* The reader should bear in mind that this is written in the mid-seventies.

4. *Editor's note:* all these texts are published in the Proceeding from which the present Brousseau text is taken.

5. Studies on this subject are currently in progress (Viennot, 1979).

6. *Editor's note:* Brousseau (personal communication) explains that his point here is that since the mathematics required for an actual proof of this equality is at too deep a level for these

students, they will at best simply accept the result, without enough understanding for it to stick with them.

7. *Editor's note:* Brousseau refers to the French curriculum of the early seventies.

8. *Editor's note:* 1n the course of 1795, the Convention (successor to the National Assembly of the French Revolution) enacted a wide range of social legislation with a considerable long-term impact. Among its issues was that of educational reform, both in terms of modifying the forms of instruction and of establishing schools.

9. More generally, all appropriate "over-teaching" has a tendency to create such obstacles; but are they avoidable?

10. We know today, thanks to A. S. Saydan (1966), that it was al-Uqlidisi who, around 952, was the first to propose the use of decimal fractions and who wrote them as we do today.

11. *Editor's note:* Such a simple thing that it doesn't merit the name "invention" [...) it easily teaches how to expedite with whole numbers without fractions all accounts encountered in the affairs of man *(English free translation).*

12. *Editor's note:* it be regulated more legitimately by the higher authorities, the aforesaid tenth division, so that everyone who wants to can use it *(English free translation).*

13. *Editor's note: toise*: vertical, graduated rod for measuring people's heights; the *pied* was equal to 32.5 cm according to the *Larousse dictionnaire de francais*, 1989.

14. *Editor's note:* Brousseau adds here: "see, for example, later chapters in this book". He is referring to texts of other authors published in the CIAEM proceedings, e.g. Ciosek, Glaeser, Janvier, Wermus, etc.

15. *Editor's note:* Brousseau, in the original version, refers the reader to a text "which is now in preparation"... see Chapter 3 and Chapter 4.

16. *Editor's note:* see Chapter 4, section 3.

17. *Editor's note:* see Chapter 4, section 2.3.

18. *Editor's note:* these examples are presented in detail in Chapter 4.

19. (Note, 1983) The second part of this remark has been shown to be false. The dialectics of the formulation and validation have been shown to be insufficient to result in teaching. A phase involving institutionalization is required.

20. *Editor's note:* see Chapter 4, section 2.2.

21. *Editor's note:* This last section is the "1983 conclusion and comments", published together with the original 1976 version in *Recherches en didactique des mathematiques* 4(2) 189-198. It follows a rather strong discussion between Georges Glaeser and Guy Brousseau about the nature of epistemological obstacles. We omit from the present translation polemical aspects too closely related to the debate of this time. See also Prelude to Chapter 2, note 2.

22. Glaeser (1981) defends the nafve use of the term "obstacle" but nevertheless refers to Bachelard, It is thus a question of a deliberately different point of view.

23. *Editor's note:* Glaeser (1981, p. 308).

24. *Editor's note:* This definition, first published in a work of a Brousseau's student, Duroux (1982), is reproduced in section 2.2. of this Chapter.

25. *Editor's note:* In his autobiography entitled "*La vie d'Henri Brulard*" (1835), Stendhal recalls his problems in making sense of the "rule of signs". His objection is that if one divides the world into two parts, one for positive numbers and the other for negative numbers, he does not see how if A and B are in the negative part, taking B A times gets you into the other part... But if Stendhal is bothered by the extension of multiplication to integers, the question of d'Alembert is of another nature: should one even consider negative numbers to be numbers? See Glaeser (1981, pp. 310-311 and 323-325).