I will refer so often to the Hilbert Euclidean parallel postulate that I hereby declare HEpp an abbreviation for that phrase!

Problem 1: (Exercise 3, page 146)

Suppose we have three points A, B, and C satisfying $A * B * C$:

(a) Prove $AB \subseteq AC$.
Prove $CB \subseteq CA$.

(b) Prove $AC \subseteq AB \cup BC$.

(c) Prove $AC = AB \cup BC$.

(d) Prove $B \in AB \cap BC$

(e) Prove B is the only point in $AB \cap BC$, thereby finishing a proof of Proposition 3.5.

Answer: Suppose $A * B * C$, which by BA 1 tells us the three points are collinear. We imagine them as in this picture, but of course we can't use any properties except those following from our axioms:

By definition (page 109) a segment such as AB is the set of those points which are between A and B together with A and B themselves. For (a) we need first to show that every point that is in AB is also in AC. If P is any point in AB, then P must be A or B or else $A * P * B$. If $P = A$ then by the definition of AC we have $P \in AC$. If $P = B$ then since we know $A * B * C$ we have $P \in AC$. If $A * P * B$, we use Proposition 3.3 (with P playing the role of B in the proposition, B the role of C, and C the role of D) and have $A * P * C$ so $P \in AC$. Hence for any point P in AB we have $P \in AC$, showing $AB \subseteq AC$. (I won't go into as much detail on subsequent problems, but showing each element of AB is an element of AC is what is essentially required whenever you are showing set inclusion $AB \subseteq AC$.)

Now for the second part of (a): The argument in the previous paragraph is perfectly symmetric (betweenness $A * B * C$ doesn't say anything about which one is on the left or right!) so exactly the same argument but swapping the roles of A and C takes care of this.

For (b): From (a) we have that any point in $AB \cup BC$ (since $BC = CB$) is in AC: We now have to show the reverse. There are several ways to do this, here is one. Let P be any point in AC: By definition, $P = A$ or $P = C$ or $A * P * C$. If $P = A$ then $P \in AB$ so $P \in AB \cup BC$, and similarly for the case $P = C$. If $A * P * C$: By Proposition 3.4, P must either be in ray $\overrightarrow{BA}$ or $\overrightarrow{BC}$. If $P \in \overrightarrow{BA}$ then by the definition of a ray, $P \in AB$ or $B * A * P$. But if $B * A * P$ then using Proposition 3.3 we would have $C * A * P$, but that contradicts $A * P * C$ and BA 3. So $P \in AB$. Similarly if $P \in \overrightarrow{BC}$ we get $P \in BC$, and we are through.

For (c): Since $AC = CA$ and $BC = CB$, from (a) and (b) we have both $AB \cup BC \subseteq AC$ and $AC \subseteq AB \cup BC$. That means they must be equal.

For (d), just look at the definitions: $B \in AB$ and $B \in BC$, so $B \in AB \cap BC$.

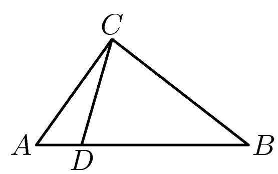
To show (e) Let E be some point off of $\overleftrightarrow{AC}$, which exists by Proposition 2.3. Let l be the line $\overleftrightarrow{BE}$. Since $A * B * C$, by definition of "sides", A and C are on opposite sides of l. If $P \in AB \cap BC$ and $P \neq B$, then P must be on the same side of l as A and also on the same side of l as C, contradicting BA 4.

Problem 2: (Exercise 9, page 148)

Given a line l and a point A on l and a point B which is not on l: Let P be any point on the ray $\overrightarrow{AB}$ other than the point A. Show that P must be on the same side of l as B.

Answer: There are many different ways to answer this. For example: If $P \in \overrightarrow{AB}$, then by definition $P \in AB$ or $A * B * P$. If $P = B$ it is certainly on the same side of l as B. If $P \in AB$ and $P \neq A$ and $P \neq B$, then $A * P * B$ so by definition of sides we have P on the same side of l as B. If $A * B * P$ then B is on the same side of l as P also, and since the "same sides" relation is symmetric we are through.

Problem 3: (Exercise 8, page 194)
We are supposed to show that in neutral geometry the greater side in a triangle is opposite the greater angle, and vice versa. The book has an outline of a proof in the statement of the problem, and I asked you to fill in the details.
Answer: We consider a triangle ΔABC. Suppose $AB > BC$. Remember these are in terms of segment and angle ordering (pages 124 and 128) not measurement. We want to show $\sphericalangle ACB > \sphericalangle CAB$. By the definition of segment ordering, there must be a point D between A and B such that $BD \cong BC$.



Then ΔBCD must be isoceles, so by Proposition 3.10 its base angles $\sphericalangle BCD$ and $\sphericalangle BDC$ must be congruent. By the Exterior Angle Theorem 4.2, $\sphericalangle BDC > \sphericalangle CAD$. By the above congruence we now have $\sphericalangle BCD > \sphericalangle CAD$. But if we use "+" as shorthand for angle addition as described in Proposition 3.19, $\sphericalangle ACB \cong \sphericalangle BCD + \sphericalangle ACD$, and then by the definition of $<$ for angles we have $\sphericalangle ACB > \sphericalangle CAD$, and $\sphericalangle CAD \cong \sphericalangle CAB$ since $\overrightarrow{AD} = \overrightarrow{AB}$, so $\sphericalangle ACB > \sphericalangle CAB$ and we are through.
Now for the other direction: Assume $\sphericalangle ACB > \sphericalangle CAB$, and we want to show $AB > BC$. We can actually use the proof in one direction that we already did to get the other using an RAA argument! Suppose the result is false, so either $AB \cong BC$ or $AB < BC$. If $AB \cong BC$, then ΔABC is isoceles so $\sphericalangle CAB \cong \sphericalangle ACB$ (by Proposition 3.10), which contradicts our assumption $\sphericalangle ACB > \sphericalangle CAB$ (using trichotmoty of angle ordering, Proposition 3.21). If $AB < BC$ then using what we have already proved, $\sphericalangle CAB > \sphericalangle ACB$, and again using trichotomy that contradicts our RAA assumption. So either case leads to a contradiction, and we are through.

Problem 4: (Exercise 10, page 195)
The exercise has two parts: Prove Proposition 4.7, and then deduce another conclusion from it, that either of the two properties that Proposition 4.7 shows equivalent is in turn equivalent to transitivity of parallelism.
Answer: First we prove the proposition, i.e. that HEpp is equivalent to "if a line intersects one of two parallel lines, then it also intersects the other."
Assume HEpp and suppose that lines l and m are parallel, and line n intersects m. We need to show that n must intersect l. Since n intersects m there is one, and only one, point P which is on both. Since m is a line through P parallel to l and HEpp says there is at most one such line, n is not parallel to l. Then n must meet l, so we are through.

Now in the other direction, assume "if a line intersects one of two parallel lines, then it also intersects the other." We need to show HEpp, i.e., there is at most one line through a given point not on a line which is parallel to that line. Suppose point P is not on line l. If m is some line through P that is parallel to l, let n be any other line through P. We need to show n cannot be parallel to l. But we have two parallel lines l and m, and a third line n that meets one of them (line m, at P). So by the assumption, n meets l, hence n is not parallel to l, finishing the proof of Proposition 4.7.

Now we need to show equivalence to "transitivity of parallelism", i.e. $l \parallel m$ and $m \parallel n$ implies $l \parallel n$. (I will ignore the trivial case $l = n$.) First lets show that "if a line intersects one of two parallel lines, then it also intersects the other" implies transivity of parallellism: Suppose $l \parallel m$ and $m \parallel n$, and (RAA) that l is not parallel to n. Then n is a line meeting one of two parallel lines l and m so n must meet the other, i.e. n meets n, but that contradicts $m \parallel n$. So the assumption l not parallel to n leads to a contradiction, it must be that $l \parallel n$, as desired.

Now in the other direction, assume transitivity of parallelism. Suppose HEpp were false, i.e. there were two lines m and n both parallel to a line l and passing through a point P not on l. Then we would have $m \parallel l$ and $l \parallel n$, so $m \parallel n$, but m and n have P in common so they cannot be parallel. So HEpp must be satisfied.

Problem 5: (Exercise 13, page 195)

The book says "prove Proposition 4.10".

Answer: First, assume HEpp. Suppose $k \parallel l$, $m \perp k$, and $n \perp l$: Suppose $m \neq n$. By Proposition 4.9, $m \perp l$. Then m and n are each perpendicular to l, so by Corollary 1 to the Alternate Interior Angle theorem (page 163) $m \parallel n$.

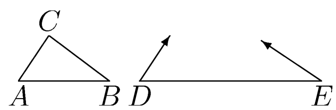
Now, in the other direction: Assume that whenever $k \parallel l$, $m \perp k$, and $n \perp l$, it follows that either $m = n$ or $m \parallel n$: We need to show HEpp. So assume that l is some line, P a point not on l, and we need to show there is at most one line through P parallel to l. We know (Corollary 2, page 163) that there is at least one line m through P parallel to l. Suppose that n is a line through P parallel to l: We must show $m = n$. Let t be the perpendicular to l through P, with foot Q on l. So far the picture looks like Figure 4.11 on page 174. Nowlet s be the perpendicular to n through P. Since t is $\perp$ to l, at Q, and s is $\perp$ to n, at P, and $l \parallel n$, by our assumption either $s = t$ or $s \parallel t$. But P is on both t and S, so they can't be parallel, so $s = t$. But by CA 4 the ray (and hence line) through P perendicular to t is unique, and both m and n go through P perpendicular to t, so $m = n$ and we are through.

Problem 6: (Exercise 3, page 229)

(a) The text, on pp. 216-217, proves that Wallis' Postulate implies the Euclidean parallel postulate: We have to prove the opposite implication. (Note that this definitely assumes we can measure angles. The statement of Wallis' Postulate (page 216) requires "similarity" of triangles, defined on page 215, which does not use measurement, but Euclid's fifth postulate does.)

Answer: We have to prove that if Euclid's fifth postulate holds, then Wallis' postulate must also. Assume Euclid's fifth postulate and assume we have a triangle ΔABC and a segment DE which is congruent to AB. We need to construct a triangle DEF which is similar to ΔABC. We can construct rays emanating from D and E such that $\sphericalangle A \cong \sphericalangle D$ and $\sphericalangle B \sphericalangle E$ by CA 4: you can pick either side of $\overleftrightarrow{DE}$, I have drawn them on the upper side in the picture on the next page.

Now we need to know that those two rays actually meet, i.e. the triangle "closes up". But the rays lie in lines met by transversal $\overleftrightarrow{DE}$, and the interior angles on that side are congruent to the angles at A and B in the triangle we started with so by Corollary 2 to the EA Theorem (page 171) the sum of those two angles is less than $180°$. So by Euclid's fifth postulate, the two rays do meet: call the point where they meet F. Now we know the angles at D and A are congruent, and



the angles at E and B are congruent. But if we have the Euclidean fifth postulate we also have that any triangle has angle sum equal to $180°$. So the angle at C must have measure $180-$the sum of the angles at A and B, which is the same as $180-$the sum of the angles at D and E, which has to be the measure of the angle at F, so the angles at C and F are also congruent and we have $\Delta ABC \sim \Delta DEF$ as desired.

(b) We have to show that a modified version of Wallis' Postulate, where we replace similarity by congruence, must be true, not needed as an additional assumption, in any neutral geometry.

Answer: If we start with ΔABC and we have a segment DE which is congruent to AB, we need to show there is a triangle ΔDEF which is congruent to ABC. But that is exactly the Corollary to SAS on page 122, which was proved in neutral geometry.