Mathematics 461 (Wilson) Fall semester 2008

Midterm exam October 29, 2008 ANSWERS

Problem 1
Prove that a line cannot be contained in the interior of a triangle.
Answer:
Let $l = \overleftrightarrow{DE}$ be any line and ΔABC any triangle.

1. (RAA) Assume l is contained in the interior of ΔABC, i.e every point on l is in the interior of ΔABC.
2. Then D and E are in the interior of ΔABC. (By step 1)
3. Hence the ray $\overrightarrow{DE}$ emanating from D meets ΔABC in one of its sides, by Proposition 3.9.
4. Thus there is a point F on $\overrightarrow{DE}$ which is also on one of the sides of ΔABC. (Just naming the point whose existence was found in step 2)
5. But every point of $\overrightarrow{DE}$ is also a point of $\overleftrightarrow{DE}$, by Proposition 3.1, so F is on l. (Step 3)
6. But if F is on one of the sides of ΔABC then F is not in the interior of ΔABC. (Definition of the interior)
7. So there is a point F on l which is not in the interior of ΔABC, contradicting the RAA assumption, and we are through.

Here is another proof I did not think of, from one of the papers, that is much shorter!
Suppose line l is entirely contained in the interior of ΔABC. Then l never intersects $\overleftrightarrow{AB}$ and so l is parallel to $\overleftrightarrow{AB}$. Similarly l is also parallel to $\overleftrightarrow{AC}$. But parallelism is transitive: If l is parallel to m and m is parallel to n then l is parallel to n. (This was stated in the student's proof...How would you justify it?) So $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ must be parallel, each is parallel to l, but that is contradicted by the fact that A is on each of them.

Problem 2
We call a set of points S *convex* if, whenever A and B are in S, the entire segment AB is contained in S.
Prove that a half plane (all of the points on the same side of some given line) is a convex set.
Answer:

1. Let l be a line and let S be one of the half planes produced by l. (Just giving names to the things we need to talk about.)
2. Let A be some point in S. (Again just naming: If there were no points in S it would be convex trivially so there is no loss in assuming there is a point in S.)
3. If B is some other point in S, consider the segment AB. (Giving a name to what we need to talk about in the definition of convexity. If there are no other points than A in S, S is trivially convex.)
4. AB has no points on l. (Definition of "same side" together with the fact that $B \neq A$.)

5. If C is any point on AB other than A and B (which are already known to be in S), C is on the same side of l as A. ($A * C * B$, so every point of AC is a point of AB (Proposition 3.5), and then apply definition of "same side")
6. So each point of AB is in S, hence by the definition S is convex.

Problem 3
Prove Proposition 2.3. (The proposition says "For every line, there is at least one point not lying on it.")
Answer: There are several ways to do this. I think this is the shortest:
Suppose l is a line and there are no points that are not on l. Then for any three points P, Q, R, they must all be on l. That violates Incidence Axiom 3.
(Note this did not assume P, Q, R were distinct, or even that there are 1, 2, or 3 points.)

A longer proof:
Let l be any line. By proposition 2.2 there are lines m and n such that no point is on all three lines l, m, and n. If $m = l$ and $n = l$ then every point on l is on all three lines (there are points on l by Incidence Axiom 2) contradicting the previous sentence, so at least two of l, m, and n are not the same. We assume without loss that $m \neq l$. Case (i): m is parallel to l. Then there is some point P on m (by Incidence Axiom 2) and since m is parallel to l, P must not be on l, and we are through. Case (ii): m is not parallel to l. Then there is some point P on both l and m. By Incidence Axiom 2 there are at least two distinct points on m, so there must be a point Q on m with $P \neq Q$. If Q were on l, then both l and m must be the unique line through P and Q (Incidence Axiom 1), but we know $m \neq l$, so Q is not on l and we are through.

Problem 4
Justify each step in the following proof of Proposition 3.11: (The proposition says "If $A * B * C$, $D * E * F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.")

(1) Assume on the contrary that BC is not congruent to EF.
Reason: RAA assumption

(2) Then there is a point G on $\overrightarrow{EF}$ such that $BC \cong EG$.
Reason: Congruence Axiom 1

(3) $G \neq F$.
Reason: RAA: If $G = F$, then $EF = EG$ and $EG \cong BC$, so $EF \cong BC$ by Congruence Axiom 3, but that contradicts assumption (1).

(4) Since $AB \cong DE$, adding gives $AC \cong DG$.
Reason: $AB \cong DE$ is given. Addition uses Congruence Axiom 3.

(5) However, $AC \cong DF$.
Reason: Given.

(6) Hence $DF \cong DG$.
Reason: Congruence Axiom 2.

(7) Therefore, $F = G$.
Reason: Uniqueness of G in Congruence Axiom 1, as used in step (2).

(8) Our assumption has led to a contradiction; hence, $BC \cong EF$.
Reason: Our assumption was that the two segments are not congruent, but that led to both $F \neq G$ (3) and $F = G$ (8), so the assumption must be false.

Problem 5

Prove Proposition 3.6. (The proposition says "Given $A * B * C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$".)

Answer: First we prove "Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$":

The definition of a ray tells us that $\overrightarrow{BA}$ consists of the segment BA together with the points P such that $P * A * B$, and similarly $\overrightarrow{BC}$ is the union of BC and $\{P|B * C * P\}$. So if P is any point in both $\overrightarrow{BA}$ and $\overrightarrow{BC}$, we can consider four separate cases.

(i) $P \in BA$ and $P \in BC$

(ii) $P \in BA$ and $B * C * P$

(iii) $P * A * B$ and $P \in BC$

(iv) $P * A * B$ and $B * C * P$

Using Betweenness Axiom 1 and swapping the roles of A and C turns case (ii) into case (iii), so we only need to prove the desired result for cases (i), (ii), and (iv).

Case (i): This is exactly the situation described in Proposition 3.5, so P must be B.

Case (ii): If $P \in BA$, then $P = B$, in which case we are through, or $P = A$, or $A * P * B$. (Definition of a segment) We are given $A * B * C$ so in the second and third of these cases we have $P * B * C$ using Proposition 3.3. But we cannot have both $P * B * C$ and $B * C * P$, by Betweenness Axiom 3. So only $P = B$ remains.

Case (iii): $A * B * C$ (given) and $B * C * P$ imply $A * B * P$ by Proposition 3.3. But $A * B * P$ and $P * A * B$ are incompatible by Betweenness Axiom 3. So this case cannot exist.

So at this point we know that no point other than B could be in both $\overrightarrow{BA}$ and $\overrightarrow{BC}$, but by the definitions of those rays B is in both, so B is the only point on both, as desired.

We still have to prove the second claim, that $\overrightarrow{AB} = \overrightarrow{AC}$:

By definition $\overrightarrow{AB} = AB \cup \{P|A * B * P\}$, and $\overrightarrow{AC} = AC \cup \{P|A * C * P\}$

If $P \in \overrightarrow{AB}$ then either $P \in AB$ or $A * B * P$. If $P \in AB$, then since AB is a subset of AC by Proposition 3.5, $P \in \overrightarrow{AC}$. If $A * B * P$, P might be C, but that is certainly in $\overrightarrow{AC}$. If $P \neq C$, then either $B * P * C$ or $B * C * P$: The first makes $P \in BC$ which is a subset of AC so $P \in AC$ and hence $P \in \overrightarrow{AC}$. The second together with Proposition 3.3 makes $P \in \{P|A * C * P\}$ so again $P \in \overrightarrow{AC}$. Thus we have shown $\overrightarrow{AB} \subseteq \overrightarrow{AC}$.

Now we need to show the reverse inclusion, $\overrightarrow{AB} \supseteq \overrightarrow{AC}$, to show they are equal.

If $P \in \overrightarrow{AC}$ then either $P \in AC$ or $A * C * P$. The second, with Proposition 3.3, gives $A * B * P$, so $P \in \overrightarrow{AB}$. If $P \in AC$, then $P = A$ or $P = C$ or $A * P * C$. If $P = A$, $P \in AB$, so $P \in \overrightarrow{AB}$. If $P = C$ then $A * B * P$ so $P \in \overrightarrow{AB}$. If $A * P * C$, then $A * P * B$ or $P = B$ or $B * P * C$. If $A * P * B$ or $P = B$ then $P \in AB \subseteq \overrightarrow{AB}$. If $B * P * C$ then $A * B * P$ by Proposition 3.3, so again $P \in \overrightarrow{AB}$. That completes the proof.