Math 340 Lectures 1 and 2, Spring 2011 — Wilson

Topics for final exam
(Exam: 7:45–9:45 AM, Thursday, May 12, 145 Birge Hall)

The format and rules for the exam will be similar to the midterm exams:

You are allowed to use a calculator, including graphing calculators, but not a portable computer or any other device that can browse the web. But there are several things you can infer from access to a calculator: (a) I will try to make the exam pretty "calculator neutral", so that someone with an expensive calculator has not bought much of an advantage over someone who does not have that. (b) In grading we will be concentrating on your showing how you get your answers rather than the numbers making up those answers. So be sure to show your work and to explain why you are doing things. Don't put in lots of extra words just to make the answer longer, but do make clear what you are doing.

You are allowed to bring in "notes": You can use up to two letter-sized sheets of paper. On those you can write anything you want, you can computer print it in tiny letters, use both sides, what ever.... But I suggest you limit yourself to specific things you believe you might get confused on: If you try to reconstruct the whole course in these notes, you will spend the entire exam time searching for things.

The exam will have N problems, for some $10 \le N \le 12$, and there will be space on the exam both for you to write your answers and for some scratch work. (We will have some extra paper in case you need more space for either purpose.) So all you need to bring is something to write with and your knowledge, and a calculator if you choose.

There will be some "proof" problems, but less of a percentage on the final will be about proof than was on the second exam. The exam will be "cumulative": That is unavoidable in a math class, you cannot do problems on more recent material without using earlier skills the recent material built on, so my saying so really does not tell you anything. But it will also have a sprinkling of problems from the whole semester, with slightly greater emphasis on material that has not been covered on earlier exams. (This way you could show us that you have now mastered material that you had trouble with on an earlier exam.) For more specific information as to coverage, refer to the list below. I have put in less information here on early material, since the descriptions of the first two exams as well as the exams themselves (with and without answers) are still posted at our class website, http://www.math.wisc.edu/ wilson/Courses/Math340/340spr11/index.htm.

§1.1-§1.5: You surely need to know about matrices, matrix operations, row reduction, and solving equations. But since those are implicit in later material such as finding eigenvectors, there will not be problems directed explicitly at this material.

§1.6: Matrix Transformations: We did not include this on earlier exams because it was going to be repeated in Chapters 6 and 7. There will of course be problems on those chapters now, and reviewing §1.6 might clarify some of the recent work.

§2.1-§2.4: Again most of this is implicit in later work. But in addition to <u>using</u> the material on echelon forms and solving linear systems, there could be specific problems asking you to find the inverse of a matrix or to express a (nonsingular) matrix as a product of elementary matrices. There will be no problems on §2.4.

§3.1-§3.4: Of course we have to compute determinants in order to find the characteristic polynomial of a matrix, so the various tools in these sections could be essential. There will not be problems specifically on permuatations or their signs, or on Cramer's rule. There could be problems involving cofactor expansions or the adjoint of a matrix.

§4.1: There will be no problems specific to this section, but being able to work in $\mathbb{R}^2$ and $\mathbb{R}^3$ is essential for almost everything since this point.

§4.2-§4.3: You certainly need to know what vector spaces and their subspaces are, definitely not restricted to $\mathbb{R}^2$ or $\mathbb{R}^3$! You should be able to answer questions like "Is this a vector space?" or "Is this a subspace?", with reasons. Because time is limited, I won't ask you a question that requires verifying all of the requirements (eight written out, plus the two amounting to closure of $\oplus$ and $\odot$) in the definition of a vector space, but you should definitely know them as I might ask for some of them. You should know and be able to prove the properties in Theorem 4.2. (And of course you should realize that a theorem about vector spaces does not assume they are column or row vectors of numbers!)

§4.4-§4.7: You definitely need to know what it means for vectors to span something, and how to tell if some vector is in the span of others. Likewise you need to be able to tell if a set of vectors is linearly dependent or independent. Combining these, you should understand what a basis is. For some spaces it is reasonable to ask you to find a basis, e.g. for the space of all solutions to $A\vec{x} = \vec{0}$ where A is some given matrix. Or if you are given a basis, you should be able to find coordinates with respect to that basis for some particular vector. Theorem 4.8 justifies this calculation. You might be asked to find a basis consisting of some of the vectors in a spanning set, or to extend a linearly independent set to a basis, i.e. carry out what happens in the proof of Theorem 4.11. You should be able to find the dimension of an appropriate space.

§4.8: You should be able to find the matrix (the matrix the book calls $P_{S \leftarrow T}$) that effects change of coordinates as the basis changes.

§4.9: Be able to use Theorem 4.19: It can save quite a bit of work! Know the several statements about rank that are embodied in Theorem 4.20 and the following corollaries, and the equivalent properties listed on page 281.

§5.1-§5.4: You need to know what an inner product is, both for the special case of the dot product in $\mathbb{R}^n$ and for the general case of any function from pairs of vectors to numbers that satisfies Definition 5.1. But, in the interests of time, any problem requiring calculation with inner products will use the dot product. For example, a problem asking you to carry out the Gram-Schmidt process would use $\mathbb{R}^n$ and the dot product rather than an inner product requiring computation of integrals. There will be no problems relating to §5.2.

§6.1: You should be able to tell if some given function is a linear transformation, and know what Theorem 6.1 tells us. We have had several different settings in which the real content of Theorem 6.2 has appeared, summarized as "what happens to a basis is enough to tell you what happens to anything" for all sorts of linear operations.

§6.2: You should be able to find both the kernel and the range, and the dimension of each, for a linear transformation. You should be able to prove that each is a subspace of its respective vector space. You should be able to use Theorem 6.4 to test whether a linear transformation is $1-1$, and Definition 6.4 to tell if it is onto. In particular note the interaction with Theorem 6.1 in the previous section: If L is a linear transformation, it always takes the zero vector in one space to the zero vector in the other. The relevant question for whether the transformation is $1-1$, i.e. whether its kernel is just $\vec{0}$, is whether the zero vector in the first space is the only vector it takes to the zero vector in the second space. Note that Theorem 6.6 is a restatement of Theorem 4.19 in a more general setting. There will not be any problems dealing with the inverse of a linear transformation.

§6.3: You should be able to find the matrix representing a linear transformation with respect to given bases.

Note that there are always two bases involved, but the second might be the same as the first if the two spaces are the same. E.g. in problem 1(a) on page 397, where the spaces are $\mathbb{R}^2$ for both inputs and outputs of the function L, it would really be better to say "Find the representation of L with respect to S and S." instead of "...with respect to S." Phrased this way, the matrix always is produced by applying the linear transformation to each vector in turn from the basis for the first space, finding the coordinates of the result with respect to the basis for the second space, and using those as the columns of the matrix.

§7.1: You should understand what eigenvectors represent in a geometric way, vectors that have the same or opposite direction after applying a linear transformation. (Phrased this way you can see why we never accept $\vec{0}$ as an eigenvector, since it does not determine a direction.) You should be able to compute eigenvalues and the corresponding eigenvectors: You will not be expected to find roots of particularly complicated polynomials, but you should be able to determine the roots for any degree two polynomial and for degree three if there is an "obvious" factor. Be sure you know some way to express all of the infinitely many eigenvectors going with an eigenvalue: It might be good to go back and review how to find a basis for the set of solutions to a set of homogeneous equations.

§7.2: You should know what it means for $n \times n$ matrices to be similar, that similar matrices have the same eigenvalues, and how easy it is to find eigenvalues and corresponding eigenvectors for a diagonal matrix. (Hence why it is so nice, starting from some $n \times n$ matrix A, when we can find a diagonal matrix D that is similar to A.) But you will not have to find the matrix P such that $D = P^{-1}AP$. You should know that eigenvectors going with distinct eigenvalues are always orthogonal, and that A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ composed of eigenvectors of A.