Math 340 Lectures 1 and 2, Spring 2011 — Wilson

Topics for second midterm exam,
(5:30–7:00 PM, Thursday, April 14, in B130 Van Vleck)

The general nature of the exam will be similar to that of the first exam. I do not yet know exactly how many problems will be on this exam. Again the problems listed on the class syllabus are a good guide to what can appear on the exam, except that I am well aware that some take too long to be reasonable exam questions. I have comments below on that, and of course in general I might ask something that amounted to just part of a homework problem in order to limit the required time. Although the original syllabus said this exam would go through §5.4, we will actually only have material through chapter 4.

Material that was on the first exam, which went through §3.3, will not be explicitly asked about on this exam, but as usual in math and science you really cannot do later material without understanding what came earlier. So if there was something on the first exam that either you messed up then or you think now you may have forgotten, it would be good to review that. The first exam, both with and without answers, is still on our class website, http://www.math.wisc.edu/∼wilson/Courses/Math340/340spr11.

As before, you are allowed to use a calculator, including graphing calculators, but not a portable computer or any other device that can browse the web.

You are allowed to bring in "notes" amounting to either two "note cards", each not bigger in area than one half of an ordinary sheet of typing paper, $8\frac{1}{2}$ inches by $5\frac{1}{2}$ inches (a large index card, 6×8 inches, would be OK), or one ordinary sheet of paper. (Hence in particular you could use your notes from the first exam together with a second card of comparable size if that were convenient.) On these cards you can write anything you want, you can computer print in tiny letters, use both sides, whatever. . . .

Here are section-by-section comments. I point out that sections 4.6 and 4.8 have been the focus of much of our attention, and that could signal their importance for the exam!

- §3.4 Inverse of a Matrix: There are two different kinds of things that matter here (and with determinants). One is how to compute the inverse, efficiently enough that you could find an inverse in the time and under the pressures of an exam. The other is what the inverse "does", so that things like "if $A\vec{x} = \vec{b}$ then $\vec{x} = A^{-1}\vec{b}$" have become natural and don't take a lot of time to decode. Also related to how expansion by cofactors "works" is the distinction between getting A^{-1} if you use cofactors and multipliers from the same row (or column) and getting 0 if you use "alien" cofactors, i.e. multipliers and cofactors consistenly from different rows (or columns). That fact is why A and its adjoint interact as they do. This section also takes you back (e.g. in Theorem 3.12) to basic terminology and special matrices, in this case a diagonal matrix. A problem like number 9 on page 169 pulls all this together.

- §3.5 Other Applications of Determinants and §3.6 Determinants from a Computational Point of View: We did talk briefly about §3.5 but there will not be exam problems on either of these sections.

- §4.1 Vectors in the Plane and in 3-Space: All of this was supposed to be review of stuff you already knew, just setting the stage for our definition of a vector space in the next section. There will be no problem directly on this section, but of course the spaces $\mathbb{R}^n$ and $\mathbb{R}_n$ have been consistently important all along as "familiar" examples of vector spaces.

- §4.2 Vector Spaces: You of course need to know what a vector space is! I would not on the exam ask you to show some set with operations was a vector space, covering all 10 (including closure) properties in the definition, just because it is too lengthy. But there could well be a problem that said you could assume certain of the properties were already proved and asked you to complete the job. You should also know the properties that are true in any vector space, that are listed in Theorem 4.2: Some parts of that theorem were left to be proved as exercises, and proving any of the parts could be an exam problem. In particular you should remember that we consistently showed that some vector was $\vec{0}$ by showing it did what $\vec{0}$ has to do and using the uniqueness of that vector, but a problem could also ask you to show that very uniqueness. (I.e. that if a vector works like the zero vector it must be the zero vector.)

I have also many times urged you to build a "tool kit" of examples of vector spaces: That becomes critical in the next section, but this section has many such examples.

§4.3 Subspaces: You should know both what a subspace of a vector space is supposed to be (a subset that is a vector space in its own right, using the same operations) and how to show a subset is a subspace. The first idea would require you to show that all 10 of the defining requirements for a vector space were met, but Theorem 4.3 justifies doing a lot less work! Using that theorem is the most obvious use for the "tool kit" mentioned in connection with the previous section.

§4.4 Span: Linear combinations of vectors have been extremely important from this point on. Be sure you know both what it means to talk about the span of a set of vectors (and why that will always be a subspace) and how to find what the span of some particular set of vectors is: In a vector space in general it might be messy to pin down what the span of given vectors was, but in this section are good examples for some familiar spaces. (Note that the "row space" and "column space" of a matrix, just now encountered in §4.9, are examples of spans of particular sets of vectors.)

§4.5 Linear Independence: Again this hits on linear combinations. With regard to the span, the question was what can you get by using linear combinations. For linear independence, the question is whether something that you can get could actually be produced in more than one way. Be sure you know how to show vectors in some space are, or are not, linearly independent!

§4.6 Basis and Dimension: Of course you need to know the details, both the facts (theorems and corollaries) and the definitions, from this section. It may help to try to remember the big picture: If I start with a small set of vectors in some space, that might be linearly independent but not enough to span the whole space. If I add carefully chosen vectors I can increase the span while maintaining linear independence. When I get to a set that does span the whole space, there is nothing more I can add without losing linear independence: At this threshold we have a basis. Or, I could start with a set of vectors large enough to span the whole space, but they might not be linearly independent. In that case I can select carefully some vector to remove that leaves a set spanning the whole space. If that set is not linearly independent, I can remove another, and repeat, until I get to a set that is "just big enough" to span the space: If there is nothing more I can remove, the resulting set will be linearly independent so we have come to the same boundary case, a basis, from the other direction.

§4.7 Homogeneous Systems: Except for the very last part (beginning on page 250) this section is all about the set of solutions of $A\vec{x} = \vec{0}$ as a vector space, a subspace of $\mathbb{R}^n$ if A is $m \times n$. That space, the solution space or null space, is worth understanding as a vector space where some of the problems like "find a basis for this space" and "what is the dimension of this space" have concrete, algorithmic, solution techniques. It also comes back to haunt us in §4.9.

§4.8 Coordinates and Isomorphisms: We have spent an enormous amount of time on this section. You should certainly be fluent with the coordinates of a vector with respect to some given basis, and you should be able to find the matrix that tells how coordinates change when we change to a different basis. The abstract definitions of "linear transformation" and "isomorphism" won't themselves appear in problems on the exam (but they certainly will when we get to chapter 6,) but what is critical for a linear transformation, the fact that it "works right" with linear combinations, is fundamental to this whole section.

§4.9 Rank of a Matrix: You should know what the row and column spaces for a matrix are and how to find bases for them. There are two kinds of problems frequently seen here: (a) Find a basis for this space, and (b) Find a basis for this space from among some given set of vectors. Be sure to understand the difference and be able to deal with either. Going with this you need to know what the (row or column or "neither") rank of a matrix is, how to find it, and how that combines with the dimension of the solution space of $A\vec{x} = \vec{0}$ (Theorem 4.19).