

Your Name: _____

Please circle your TA's name: Jae-Ho Lee Andrew Bridy

Mathematics 340, Spring 2011

Lectures 1 & 2 (Wilson)

Final Exam May 12, 2011

There is a problem on the back of this sheet: Be sure not to skip over it by accident!

There some scratch paper at the end of this exam. If you need more scratch paper, please ask for it.

You may refer to notes you have brought on one or two sheets of paper, as announced previously.

The symbol \vec{x} means $\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$, \vec{b} means $\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$, and $\vec{0}$ means $\begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$, for whatever size is appropriate.

You may assume that vector spaces \mathbb{R}^n , \mathbb{R}_n , M_{mn} , and P_n (with their usual operations), that we have frequently used, are vector spaces, without further proof. Do not assume other spaces are vector spaces unless specifically told to do so.

BE SURE TO SHOW YOUR WORK, AND EXPLAIN WHAT YOU DID. YOU MAY RECEIVE REDUCED OR ZERO CREDIT FOR UNSUBSTANTIATED ANSWERS.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
TOTAL	200	

Problem 1 (20 points)

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$:

- (a) Find a basis for the solution space (null space) of A , the subspace of \mathbb{R}^5 consisting of solutions of $A\vec{x} = \vec{0}$.

Be specific as to what vector(s) make up your basis!

- (b) What is the dimension of the null space of A (i.e. the nullity of A)?

Problem 2 (20 points)

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$ (the same matrix as in problem 1):

(a) What is the rank of A ?

(b) Find a basis for the row space of A .

(c) Find a basis for the column space of A that consists of some columns of A .

Problem 3 (20 points)

For each of the following functions from \mathbb{R}^2 to \mathbb{R}^2 , tell whether it is a linear transformation or not and give reasons for your answer:

(a) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2b \end{bmatrix}:$

(b) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2 \end{bmatrix}:$

(c) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a^2 \end{bmatrix}:$

Problem 4 (20 points)

Let $A = \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$.

- (a) Find the characteristic polynomial of A ,
- (b) What are the eigenvalues of A ?
- (c) For each of the eigenvalues, describe all of the eigenvectors.
(I.e., express in some way all of the eigenvectors, not just an example, and on the other hand not including any vectors that should not be eigenvectors.)

Problem 5 (20 points)

Assume L is a linear transformation from a vector space V to a vector space W .

Prove that the range of L is a subspace of W .

Problem 6 (20 points)

Let L be the linear transformation from P_2 (the space of polynomials of degree at most two) to P_2 defined by $L(p(t)) = p'(t)$, the derivative of the polynomial function. Using the “standard” ordered basis $B = \{1, t, t^2\}$ (with the vectors in that order!):

(a) Find the matrix A representing L with respect to B and B .

(b) For the polynomial $p(t) = 3 - 2t + 2t^2$, what is the coordinate vector $[p(t)]_B$?

(c) What is the coordinate vector $[L(p(t))]_B$ for $L(p(t))$? (Use the same $p(t)$ as in (b).)

(d) Use the matrix from (a) and the vectors from (b) and (c) to show that the matrix “does the right thing”, i.e. that multiplying a coordinate vector by the matrix does give you the coordinates for the result of applying L .

Problem 7 (20 points)

Suppose that L is a linear transformation from a vector space V to a vector space W , and that the kernel of L contains only the zero vector of V . Show that L must be $1 - 1$.

(This is actually an “if and only if” result, but you are only asked to prove one direction. Be sure you prove the correct direction.)

Problem 8 (20 points)

For the vector space $V = \mathbb{R}^3$, with ordered bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$:

Find the matrix $P_{S \leftarrow T}$ for changing coordinates from T to S .

Problem 9 (20 points)

The set of vectors $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for \mathbb{R}^3 ,

the vector space of three element column vectors.

Using the ordinary “dot product” as an inner product on \mathbb{R}^3 :

- (a) Use the Gram-Schmidt process starting with B to find an orthogonal basis for \mathbb{R}^3 , i.e. a basis where each pair of distinct vectors is orthogonal.

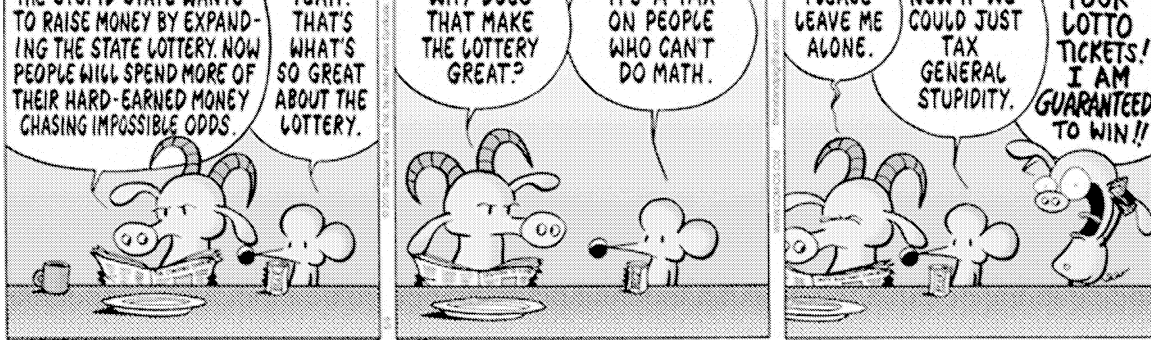
- (b) Continue from what you found in (a) to get a basis which is orthonormal, i.e. in addition to being orthogonal it now has the magnitude (norm, size) of each vector equal to 1.

(If your answer to (a) was already orthonormal, make note of that here.)

Problem 10 (20 points)

Suppose L is a linear transformation from V to W . Prove:

If L is 1-1 and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly independent set in V , then $\{L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_k)\}$ is a linearly independent set in W .



Scratch Paper

(not a command!)