Final Exam May 12, 2011 With Answers

Problem 1

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$:

(a) Find a basis for the solution space (null space) of A, the subspace of $\mathbb{R}^5$ consisting of solutions of $A\vec{x} = \vec{0}$.

ANSWER:

First we reduce A to Reduced Row Echelon Form, getting $A_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. We see that the fifth column does not contain a leading entry, so the corresponding variable x_5 can be given an arbitrary value. From the first row we see that $x_1 + x_5 = 0$, i.e. x_1 must be $-x_5$. Similarly from the remaining rows we get $x_2 = x_5$, $x_3 = -2x_5$, and $x_4 = -2x_5$. So solutions must look like $\begin{bmatrix} -x_5 \\ x_5 \\ -2x_5 \\ -2x_5 \\ x_5 \end{bmatrix}$, i.e. the solutions are all multiples of $\begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$. Hence a basis for the solution space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) What is the dimension of the null space of A (i.e. the nullity of A)?

ANSWER:

Since there was one vector in the basis, the dimension of the null space is 1.

Problem 2

For the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$ (the same matrix as in problem 1):

(a) What is the rank of A?

ANSWER:

We saw in the answer to problem 1 that the reduced row echelon form of $A_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Since there are four non-zero rows in that matrix, the rank of A is 4.

(b) Find a basis for the row space of A.

ANSWER:

From (a) We know that the row space will be a four dimensional subspace of $\mathbb{R}_5$, so we know we need four 5-element row vectors. We could use the rows of A, or the rows of A_R. Using the rows of A, one basis is $\{[1, 2, -2, 3, 1], [0, 1, 0, 0, -1], [0, 0, 1, 0, 2], [0, -1, 1, 1, 5]\}$.

(c) Find a basis for the column space of A that consists of some columns of A.

ANSWER:

We can use the columns from A_R that have leading entries to pick out columns from A: The leading entries are in the first four columns, so we use $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Problem 3

For each of the following functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, tell whether it is a linear transformation or not and give reasons for your answer:

(a) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2b \end{bmatrix}$:

ANSWER:

This <u>is</u> a linear transformation. You could check it directly from the definition, or you could write it as $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$. We know that multiplication by a matrix always gives a linear transformation.

(b) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a+2 \end{bmatrix}$:

ANSWER:

This <u>is not</u> a linear transformation. $L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so this function does not take the zero vector to the zero vector, which any linear transformation must do.

(c) $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a^2 \end{bmatrix}$:

ANSWER:

This <u>is not</u> a linear transformation. If we can find any instance where the requirements for a linear transformation fail, that would justify this claim, so there are many possible reasons to give. I note that $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, while $2L\left(\begin{bmatrix} 1 \end{bmatrix}\right) = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. So this function does not satisfy $L(c\vec{v}) = cL(\vec{v})$.

Problem 4

Let $A = \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial of A,

ANSWER:

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \lambda+1 & 4 \\ 1 & \lambda-2 \end{bmatrix}$ The characteristic polynomial of A is the determinant of that matrix, i.e. $(\lambda+1)(\lambda-2) - 4 \times 1 = \lambda^2 - \lambda - 6$.

(b) What are the eigenvalues of A?

ANSWER:

We can factor $\lambda^2 - \lambda - 6$ as $(\lambda - 3)(\lambda + 2)$, so the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

(c) For each of the eigenvalues, describe all of the eigenvectors.

ANSWER:

We substitute each value of λ into the matrix $\lambda I - A$ above and solve the corresponding homogeneous equations.

For $\lambda = 3$:
$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solutions $x_1 = -x_2$, i.e. all multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the eigenvectors are all the non-zero multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda = -2$:
$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solutions $x_1 = x_2$, i.e. all multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the eigenvectors are all the non-zero multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 5

Assume L is a linear transformation from a vector space V to a vector space W.

Prove that the range of L is a subspace of W.

ANSWER:

The range, the set of all vectors $\vec{w}$ in W such that $\vec{w} = L(\vec{v})$ for some $\vec{v}$ in V, is by definition a subset of W. We need to show (a) it is not empty, (b) it is closed under addition, and (c) it is closed under multiplication by scalars.

(a) Since $L(\vec{0}_V) = \vec{0}_W$ for any linear transformation from V to W, we have that $\vec{0}_W$ is L(something in V), so the range of L contains $\vec{0}_W$, so the range is not empty.

(b) We need to show that the sum of any two vectors in the range produces a vector in the range. Suppose $\vec{w}_1$ and $\vec{w}_2$ are any two vectors in the range of L. Since they are in the range, there must be vectors $\vec{v}_1$ and $\vec{v}_2$ in V such that $\vec{w}_1 = L(\vec{v}_1)$ and $\vec{w}_2 = L(\vec{v}_2)$. But then $\vec{w}_1 + \vec{w}_2 = L(\vec{v}_1) + L(\vec{v}_2)$, and since L is a linear transformation that must be $L(\vec{v}_1 + \vec{v}_2)$, so $\vec{v}_1 + \vec{v}_2$ is a vector in V that L takes to $\vec{w}_1 + \vec{w}_2$, hence $\vec{w}_1 + \vec{w}_2$ is in the range of L.

(c) We need to show that any scalar multiple of a vector in the range of L is in the range of L. Let $\vec{w}$ be any vector in the range of L, and let c be any scalar. Since $\vec{w}$ is in the range, $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$. Then $c\vec{w} = cL(\vec{v}) = L(c\vec{v})$ (since L is a linear transformation), hence $c\vec{w}$ is "L of something in V", i.e. $c\vec{w}$ is in the range of L.

Problem 6

Let L be the linear transformation from P_2 (the space of polynomials of degree at most two) to P_2 defined by $L(p(t)) = p'(t)$, the derivative of the polynomial function. Using the "standard" ordered basis $B = \{1, t, t^2\}$ (with the vectors in that order!):

(a) Find the matrix A representing L with respect to B and B.

ANSWER:

We apply L to (i.e. take the derivative of) each vector in B, and find the coordinate vector of the result with respect to B.

For the first vector 1 in B, the derivative gives 0 which is $0 \times 1 + 0 \times t + 0 \times t^2$, so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For the second vector t, the derivative is 1, so the coordinate vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Lastly, the derivative of t^2 is $2t = 0 \times 1 + 2 \times 2 + 0 \times t^2$, and the coordinate vector of that is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Putting these together, the matrix is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) For the polynomial $p(t) = 3 - 2t + 2t^2$, what is the coordinate vector $[p(t)]_B$?

ANSWER:

Since $3 - 2t + 2t^2$ is already written as a linear combination of 1, t, and t^2, we can read off the coordinate vector $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

(c) What is the coordinate vector $[L(p(t))]_B$ for $L(p(t)$?

ANSWER:

Taking the derivative, $L(3 - 2t + 2t^2) = -2 + 4t$, so its coordinate vector is $\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$.

(d) Use the matrix from (a) and the vectors from (b) and (c) to show that the matrix "does the right thing", i.e. that multiplying a coordinate vector by the matrix does give you the coordinates for the result of applying L.

ANSWER:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}.$$

Problem 7

Suppose that L is a linear transformation from a vector space V to a vector space W, and that the kernel of L contains only the zero vector of V. Show that L must be $1-1$.

ANSWER:

Assume that L is a linear transformation from V to W and that only $\vec{0}_V$ is in the kernel of L, i.e. $\vec{0}_V$ is the only vector in V that L takes to $\vec{0}_W$.

Then if $L(\vec{u}) = L(\vec{v})$, $L(\vec{u}) - L(\vec{v}) = \vec{0}_W$, and since L is linear that tells us $L(\vec{u} - \vec{v}) = \vec{0}_W$, i.e. $\vec{u} - \vec{v}$ is in the kernel of L. But the only vector in the kernel is $\vec{0}_V$, so we must have $\vec{u} - \vec{v} = \vec{v}_V$. Then $\vec{u} = \vec{v}$. So we have shown that whenever L takes two vectors to the same result in W, the two were really the same to begin with, i.e. L is $1-1$.

Problem 8

For the vector space $V = \mathbb{R}^3$, with ordered bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$:

Find the matrix $P_{S \leftarrow T}$ for changing coordinates from T to S.

ANSWER:

For each vector in T, we find its coordinates with respect to S. For the first vector, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we need to solve $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for a, b, and c. You can set that up as a system of equations, but we can also just see the answer: That last vector in S is exactly what we want, so $a = b = 0$ and $c = 1$, and the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Now for the second vector, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, solving to make $a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ give that vector: The vectors multiplied by a and by b have 0 in the third entry, so c must be 2 to make the third place work out. But that puts a 2 in the middle place: We fix that by making $b = -2$. So far that puts a 0 in the top position, so we let $a = -1$ and get the coordinates as $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$.

Moving to the third vector, in the same way we find $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so the coordinate vector is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

We assemble these as the matrix, getting $P_{S \leftarrow T} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Problem 9

$$\text{The set of vectors } B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3,$$

the vector space of three element column vectors.
Using the ordinary "dot product" as an inner product on $\mathbb{R}^3$:

(a) Use the Gram-Schmidt process starting with B to find an orthogonal basis for $\mathbb{R}^3$, i.e. a basis where each pair of distinct vectors is orthogonal.

ANSWER:

To make the notation match both our textbook and my online description, I will give names to the three vectors making up S, $u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

We create new vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ that are orthogonal as follows. Start by letting $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Now make $\vec{v}_2$ by starting with $\vec{u}_2$ and subtracting its projection onto $\vec{v}_1$: $\vec{v}_2 = \vec{u}_2 - \frac{(\vec{u}_2,\vec{v}_1)}{(\vec{v}_1,\vec{v}_1)}\vec{v}_1$.

We are to use the dot product for those inner products: $(\vec{u}_2, \vec{v}_1) = \vec{u}_2 \cdot \vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $= 2 \times 2 + 2 \times 1 + 3 \times 2 = 12$. In the same way we get $(\vec{v}_1, \vec{v}_1) = 2 \times 2 + 1 \times 1 + 2 \times 2 = 9$. So we want to compute $\vec{v}_2 = \vec{u}_2 - \frac{12}{9}\vec{v}_1 = \vec{u}_2 - \frac{4}{3}\vec{v}_1$. But for now all we care about is orthogonality, not magnitude, so we could instead use 3 times that result, $3\vec{u}_2 - 4\vec{v}_2$, and not have to deal with fractions: (You did not need to do that, it just makes the arithmetic easier to follow!) We get $\vec{v}_2 = 3 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

(As a check, the dot product of v_1 and v_2 is now $(2 \times -2) + (1 \times 2) + (2 \times 1) = 0$, so these are indeed orthogonal!)

Now we construct $\vec{v}_3$ by starting with $\vec{u}_3$ and subtracting its projections onto each of $\vec{v}_1$ and $\vec{v}_2$, $\vec{v}_3 = \vec{u}_3 - \frac{(\vec{u}_3,\vec{v}_1)}{(\vec{v}_1,\vec{v}_1)}\vec{v}_1 - \frac{(\vec{u}_3,\vec{v}_2)}{(\vec{v}_2,\vec{v}_2)}\vec{v}_2$. Computing those inner (dot) products: We already had $\vec{v}_1 \cdot \vec{v}_1 = 1$. $\vec{u}_3 \cdot \vec{v}_1 = -2 + 1 + 0 = -1$. $\vec{u}_3 \cdot \vec{v}_2 = 2 + 2 = 4$. $\vec{v}_2 \cdot \vec{v}_2 = 4 + 4 + 1 = 9$. So the formula above gives us $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{4}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Again we can simplify things bu using 9 times that vector for $\vec{v}_3$ to eliminate fractions, getting $\vec{v}_3 = 9 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, and again we can check that this does give 0 for the dot product with either $\vec{v}_1$ or $\vec{v}_2$. Summarizing, our new, orthogonal, basis is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$.

(b) Continue from what you found in (a) to get a basis which is orthonormal, i.e. in addition to being orthogonal it now has the magnitude (norm, size) of each vector equal to 1.

ANSWER:

We multiply each of these vectors by $1/||\vec{v}||$: Each has magnitude $||\vec{v}|| = \sqrt{9} = 3$, so the resulting vectors are $\left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$.

Problem 10

Suppose L is a linear transformation from V to W. Prove:
If L is $1-1$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a linearly independent set in V, then $\{L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k)\}$ is a linearly independent set in W.

ANSWER:

We are given that L is a linear transformation from V to W which is $1-1$, so its kernel is just $\vec{0}_V$. For the given linearly independent set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in V, consider the vectors $L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k)$ in W and suppose some linear combination $a_1L(\vec{v}_1) + a_2L(\vec{v}_2) + \cdots + a_kL(\vec{v}_k)$ gives the zero vector $\vec{0}_W$ in W. Since L is linear we can rewrite that as $L(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W$. But that says $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ is in the kernel of L, so it must be $\vec{0}_V$, i.e. $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V$, but the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent, so the coefficients $a_1 = a_2 = \cdots = a_k = 0$. Recapping, any linear combination of the vectors $L(v_1)$, $L(v_2)$,...,$L(v_k)$ that gives $\vec{0}_W$ must have all zero coeeficients, so the vectors are linearly independent.