First Midterm Exam February 24, 2011 Answers

Problem 1 (10 points)
Suppose, for some system of equations, the augmented matrix is row equivalent to

$$\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Give all solutions to the system of equations.
ANSWER:
The coefficient matrix is in REF but not RREF. We could either read the solutions from the matrix as it is, or convert it to RREF. I will do it from the matrix as it is: The last row says $x_6 = 5$ so there is no choice about that. The next-to-last row says $x_5 + 3x_6 = 4$, and we have $x_6 = 5$, so we must have $x_5 = 4 - 3 \times 5 = -11$. The third row says $x_4 + 2x_6 = -3$, so again using our value for x_6 we have $x_4 = -3 - 2 \times 5 = -13$. The third column has no leading entry, so the value for x_3 is arbitrary: Let it be some number α. The second row says $x_2 - x_3 - 2x_6 = 1$, so putting in α for x_3 and 5 for x_6 we get $x_2 = 1 + \alpha + 2 \times 5 = 11 + \alpha$. Finally, the first row says $x_1 + 2x_3 - x_6 = 2$, so $x_1 = 2 - 2\alpha + 5 = 7 - 2\alpha$.

You could also write this in vector form, $\vec{x} = \begin{bmatrix} 7-2\alpha \\ 11+\alpha \\ \alpha \\ -13 \\ -11 \\ 5 \end{bmatrix}$, for all numbers α.

Problem 2 (10 points)

Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & -4 & 0 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix}$

Find a matrix in Reduced Row Echelon Form that is row equivalent to A.
ANSWER:
There are many different sequences of row operations could choose to apply, but if done correctly the resulting matrix in RREF will be the same. Here is one way to get there:

Subtract (1 times) the first row from the third row, getting $\begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & -4 & 0 & 6 \\ 0 & -1 & 2 & 1 & 1 \end{bmatrix}$.

Multiply the second row by $\frac{1}{2}$, which gives $\begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & -1 & 2 & 1 & 1 \end{bmatrix}$.

Now add (1 times) the second row to the third: $\begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(That is in REF but not RREF.)

Finally, subtract the second row from the first and have $\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$.

as the answer.

Problem 3 (10 points)

For both (a) and (b) below, let $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

(a) How many solutions does the system of equations $A\vec{x} = \vec{0}$ have?

ANSWER:

There are several ways to proceed.

Perhaps the most abstract is: The determinant of A is $(1 \times (-2) - (-1) \times 2)$ which is zero, so this homogeneous system does not have only the trivial solution. But any homogeneous system has at least that trivial solution, so this must have more solutions, hence infinitely many. (Any system has zero, one, or infinitely many solutions.)

At the other extreme, very concrete: Each equation boils down to $x_1 = x_2$, so you can choose any value for one and give the other that same value and have a solution. Since there were infinitely many choices for the value we gave the first one, there are infinitely many solutions.

(b) Let $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

How many solutions does the system of equations $A\vec{x} = \vec{b}$ have?

ANSWER:

You could row-reduce the system and find a row that had zeros in the first two entries but non-zero in the third. Or, think about what the equations say: The first says $x_1 - x_2 = 2$. The second says $2x_1 - 2x_2 = 3$, so $x_1 - x_2 = \frac{3}{2}$. But $x_1 - x_2$ can't be both 2 and $\frac{3}{2}$, so these equations are inconsistent, there are no solutions.

Problem 4 (10 points)

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Find A^{-1}.

ANSWER:

The most straightforward way to do this at this point (we will get another way in §3.4) is to make a 3×6 matrix with A in the left three columns and I_3 in the right three columns, and row reduce that matrix.

We have $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ to row reduce.

If we add the second row to the first we get $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$.

Add 3^{rd} row to 2^{nd}, subtract 3^{rd} from 1^{st}, and get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$.

Since the first three columns are now I_3, the last three must be the inverse, i.e.

$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which you can check by multiplying by A.

<u>Problem 5</u> (10 points)
Theorem 1.1, part (b), says that matrix addition is associative, i.e.:

For any $m \times n$ matrices A, B, and C, $\quad A + (B + C) = (A + B) + C.$

Prove this fact.
<u>ANSWER:</u>
This is really an exercise in what equality of matrices means and how the sum of matrices is defined. $A + (B + C)$ and $(A + B) + C$, to be equal, (i) must be the same size and (ii) must have the same number at each entry. We check these:
We are given that A, B, and C are all $m \times n$. The definition of the sum of matrices then gives us that $A + B$ and $B + C$ are also $m \times n$, and then it gives us that $A + (B + C)$ and $(A + B) + C$ will also be $m \times n$. That takes care of (i).
Since each of $A + (B + C)$ and $(A + B) + C$ now is known to have m rows and n columns, there are $m \cdot n$ entries in each. We look at the entry in row i and column j for each, for i from 1 to m and j from 1 to n: If all of those entries agree, then we will have (ii) above.
In $A + (B + C)$ the entry in row i and column j will be (again by the definition of matrix addition) the sum of a_{ij} and the entry in row i and column j of $B + C$, which (yet again by the definition!) is $b_{ij} + c_{ij}$. So the entry in row i and column j of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$.
Now on the other side: In $(A + B) + C$ the entry in row i and column j will be the sum of the entry in row i and column j of $A + B$ with c_{ij}. The entry in row i and column j of $A + B$ is $a_{ij} + b_{ij}$. So the entry in row i and column j of $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij}$.
But $a_{ij} + (b_{ij} + c_{ij})$ is a sum of numbers, which we know we can reassociate as $(a_{ij} + b_{ij}) + c_{ij}$, so for each i and j the entries do agree and the result is proved.

<u>Problem 6</u> (10 points)

Find the determinant of $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & 4 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 3 & 5 & 2 & 6 \end{bmatrix}$.

<u>ANSWER:</u>
There are several ways we can do this, e.g. reduce to triangular form and keep track of the effect of each row operation, but I think the easiest will be to expand by cofactors either across the third row, where there is only one non-zero element, or down the last column. Using the third row: Calling the

matrix A, we want the sum of $a_{3j}A_{3j}$ for j from 1 to 4 (where A_{ij} means the cofactor going with row i and column j). Since $a_{31} = a_{32} = a_{34} = 0$, the first, second, and fourth of those four terms will be zero regardless of what the cofactors are. The cofactor A_{33} is found by (i) deleting the third row and third column and taking the determinant of the resulting 3×3 matrix to get the minor and then (ii) giving it an appropriate $\pm$ sign. The 3×3 matrix will be $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$. We need to take the determinant of that to get the minor. You could use the "draw lines across the matrix" technique since this is just 3×3, but we could also use expansion by cofactors of the first row in this matrix since that row is mostly zero. The only cofactor that will matter (the only one that won't be multiplied by zero) is the one going with the upper left entry: Deleting the first row and column gives us the 2×2 matrix $\begin{bmatrix} 4 & 0 \\ 5 & 6 \end{bmatrix}$, and the determinant of that is $4 \times 6 = 24$. The sign going with row 1 and column 1 comes from $(-1)^{1+1} = 1$, i.e. it is positive, so the cofactor is $+24$ and the determinant of the 3×3 matrix is (since its $1,1$ entry is 1) $1 \times 24 = 24$.
Now back to the cofactor of the $3,3$ entry in the original matrix: We now have the determinant of the 3×3 matrix that comes from deleting the third row and column is 24. So the cofactor going with that position is $(-1)^{3+3} \times 24 = +24$. The determinant of A will then be $0 + 0 + 2 \times 24 + 0 = 48$.

Problem 7 (10 points)
Theorem 3.4 says:

If a row (or column) of A consists entirely of zeros, then $det(A) = 0$.

Prove this, for either rows or columns: You do not need to do both, but be sure to say what it is that you are proving.
ANSWER:
I choose to do it for rows, and I will assume it is the i^{th} row of A that is entirely zero. I.e., I will prove "if A is an $n \times n$ matrix with $a_{ij} = 0$ for some fixed i and for $j = 1, 2, \ldots, n$, then $det(A) = 0$".
Assume A is such a matrix. The definition of the determinant tells us that $det(A)$ is a sum of $n!$ numbers. Each number is a product of n entries from A, one from each row and one from each column, with a sign attached that was derived from a permutation that picked out how the rows and columns were paired. Any of those products has to include one factor from the i^{th} row: No matter which column that entry comes from in the i^{th} row, the entry will be 0, since all entries in that row are 0. So the product, with one of the things being multiplied equal to 0, must be 0. So the sum of all $n!$ products is just $\pm 0 \pm 0 \cdots \pm 0 = 0$, and hence $det(A) = 0$.

Problem 8 (10 points)

The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is nonsingular.

Find a sequence of elementary matrices E_i such that $A = E_1 \cdot E_2 \cdots E_n$.

ANSWER:
We apply row operations that convert A to I_n, and keep track of the elementary matrices that effect those operations. There are many sequences of operations that you could choose, here is one. (For each elementary row operation I will let D_i be the corresponding elementary matrix, i.e. the result of

applying the same elementary row operation to I_n.)

First I will swap the first and third rows, getting $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$:

The elementary matrix is $D_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Now I will multiply the second row by $\frac{1}{2}$, which gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$,

with $D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(If we were just row-reducing and only cared about the final results, I might now combine subtracting the 1^{st} and 2^{nd} rows from the third. But we need elementary matrices, so we need to stick to <u>elementary</u> row operations. So we do those in two separate steps.)

Subtracting the 1^{st} row from the third gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$,

and corresponds to $D_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Subtracting the 2^{nd} row from the third gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and corresponds to $D_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Since those row operations transformed A to I_3, we have matrices D_i such that $D_4D_3D_2D_1A = I_3$. To get A as a product of elementary matrices we need to get the D_i's to the other side. To do that, we first find the inverses of the matrices D_i: Since D_i was an elementary matrix, its inverse will be what we get from I_3 by applying the "undoing" elementary operation. E.g., D_3 corresponded to "subtract the first row from the third", so its inverse would add the first row to the the third and $(D_3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, which I will call E_3. In the same way we get E_i by finding the inverse of D_i, for $i = 1, 2$, and 4: $E_1 = (D_1)^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E_2 = (D_2)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_4 = (D_4)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Now we can multiply the equation $D_4D_3D_2D_1A = I_3$ on the left, first by $E_4 = (D_4)^{-1}$, then by E_3, etc., to get $A = E_1 E_2 E_3 E_4 I_3$, and of course we can drop the I_3, so we are through. We could also write it all out as

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(I actually did multiply those out, it works! But since there would be many choices of row operations to convert A to I_3, there would be many possible answers, not all consisting of 4 matrices.)

Problem 9 (10 points) Prove:

If the $n \times n$ matrix A is nonsingular, then the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

has one and only one solution no matter what the numbers b_i are.

ANSWER:

We just consider the system in matrix form, $A\vec{x} = \vec{b}$. Since A is nonsingular, we can multiply each side of that equation (on the left) by A^{-1} and get $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$, and we can reassociate to get $\left(A^{-1}A\right)\vec{x} = A^{-1}\vec{b}$ or $I_n\vec{x} = \vec{x} = A^{-1}\vec{b}$. So (i) that collection of numbers $\vec{x}$ must be a solution, since we could now multiply on the left by A and get back to $A\vec{x} = \vec{b}$, and (ii) it must be the only solution since no matter what $\vec{x}$ was claimed to be a solution we could do this same set of operations and get $\vec{x} = A^{-1}\vec{b}$ as the only possibility.