Mathematics 234, Fall 2005 Lecture 2 (Wilson)

Second Midterm Exam November 17, 2005 ANSWERS

Problem 1

(a) Let $f(x, y) = x^2 + 2y^2$. Set up, but do not evaluate, an integral to compute the surface area on the graph of f over the region R which is the triangle with vertices $(0, 0)$, $(0, 2)$, and $(1, 0)$.

ANSWER:

The triangle has for its boundaries the lines $x = 0$, $y = 0$, and $y = 2 - 2x$. One way to set up the integrals would be

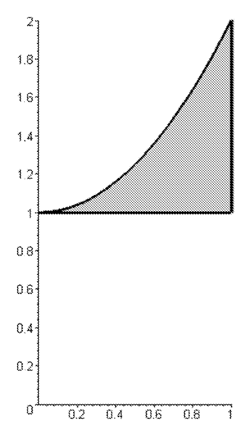
$$\int_0^1 \int_0^{2-2x} \ldots \, dy \, dx.$$

For surface area we want the integral of $\sqrt{f_x^2 + f_y^2 + 1}$. In this case $f_x = 2x$ and $f_y = 4y$. Hence we can use to calculate the surface area

$$\int_0^1 \int_0^{2-2x} \sqrt{4x^2 + 16y^2 + 1} \, dy \, dx.$$

(b) Set up the following integral with the order of integration reversed, i.e. as an integral with $dy\,dx$ replacing $dx\,dy$. Do not evaluate the integral!

$$\int_1^2 \int_{\sqrt{y-1}}^1 (e^x \cos(y)) \, dx \, dy$$



ANSWER: At the right is a sketch of the region of integration. As a $dy\,dx$ integral, we see the overall range of x values is from 0 to 1. For any x in that range, y goes from the lower line $y = 1$ up to the curved upper boundary. The curve is $x = \sqrt{y - 1}$ or $y = x^2 + 1$. Hence the integral in this form is

$$\int_0^1 \int_1^{x^2+1} (e^x \cos(y)) \, dy \, dx.$$

Problem 2

Set up an iterated integral to compute $\iiint_R (x + 2y) \, dV$,

where R is the region in space bounded by the cylinder $x^2 + z^2 = 4$, the plane $y = 0$, and the plane $y + z = 2$.

You did not have to evaluate this integral, but you were offered 5 extra-credit points for correctly evaluating it.

ANSWER: We can set this up either in rectangular or cylindrical coordinates.

In rectangular coordinates: Since the figure has straight walls in the y direction, it is simplest to put the dy integration first (i.e. on the inside). The cylinder has radius 2 and the y-axis as its center line. The plane $y + z = 2$ cuts across it, slanting down ($z = 2 - y$) as you go out in the y direction. The widest part of the (ellipse) where the plane and cylinder meet, in the x direction, is in the x-y plane where $z = 0$: Picture the circle $x^2 + z^2 = 4$ in the plane $y = 0$, and x extends from -2 to 2. If we put dx on the outside, this will give the range for the corresponding integral. If we put dz next, moving inward, the same circle shows that z ranges from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$. Now for any x and z corresponding to a point in that circle, the range of y values is from $y = 0$ at the x-z plane out to the angled plane $y = 2 - z$. Thus we can set up the integral as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{2-z} (x + 2y)\, dy\, dz\, dx.$$

Evaluating this integral is rather messy. Doing the inside integral we get $x(2 - z) + (2 - z)^2$. Integrating that in the next stage gives $\frac{2}{3}(4 - x^2)^{3/2} + 4x(4 - x^2)^{1/2} + 8(4 - x^2)^{1/2}$. Finally, evaluating the dx integral, we get 20π.

In cylindrical coordinates: The figure as given does not lend itself to cylindrical coordinates. But if we would rotate it "upward", so that the line down the center of the cylinder becomes the z-axis, then cylindrical coordinates work well. To do this we can just interchange the y and z coordinates. We must remember to do this consistently, including in the integrand $x + 2y$. The cylinder becomes $x^2 + y^2 = 4$, or $r = 2$ in cylindrical coordinates. The plane was $y + z = 2$ and swapping y and z does not affect that: In cylindrical coordinates it is $r\sin(\theta) + z = 2$. The other plane, $y = 0$, becomes $z = 0$. Remembering to change $x + 2z$ to $r\cos\theta + 2z$ and to change $dx\, dy\, dz$ to $r\, dr\, d\theta\, dz$, we have

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{2-r\sin\theta} (r^2\cos\theta + 2rz) dz\, dr\, d\theta.$$

Evaluating this integral (fortunately!) gives the same result, 20π.

Problem 3

(a) For $f(x, y, z) = x^2 + ye^z$, find the gradient field $\overrightarrow{\nabla} f$.

ANSWER: $grad(f) = \overrightarrow{\nabla} f = f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k} = 2x\vec{\imath} + e^z\vec{\jmath} + ye^z\vec{k}$

(b) Let $\overrightarrow{F}(x, y, z)$ be the vector field $2x\vec{\imath} + e^z\vec{\jmath} + ye^z\vec{k}$.

(i) Find $div(\overrightarrow{F})$.

ANSWER: $div(\overrightarrow{F}) = M_x + N_y + P_z$ where $M = 2x$, $N = e^z$, and $P = y\,e^z$. Hence $div(\overrightarrow{F}) = 2 + 0 + y\,e^z = y\,e^z + 2$.

(ii) Find $\overline{curl(\overrightarrow{F})}$.

ANSWER: We could compute this directly. But if we compare $\overrightarrow{F}$ to the answer to part (a), we see that $\overrightarrow{F} = grad(f)$. Since $curl(grad(f))$ is zero for any f, the answer is $\vec{0} = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$.

Problem 4

Find all local maxima, local minima, and saddle points, for $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$. Be sure to give both the points at which f takes on the values and the values it takes on there.

ANSWER: There is no boundary specified and f as a polynomial is continuous everywhere, so the only candidates are points where the partial derivatives are both zero. Computing, $f_x = 12x - 6x^2 + 6y$ and $f_y = 6y + 6x$. If $f_y = 0$, $6y + 6x = 0$, i.e. $y = -x$. Substituting that in $f_x = 0$ we have $12x - 6x^2 - 6x = 0$, so $6x - 6x^2 = 6x(1 - x) = 0$. Hence $x = 0$ or $x = 1$. If $x = 0$, $y = -x = 0$, while if $x = 1$, $y = -x = -1$. Thus the two candidate points are $(0, 0)$ and $(1, -1)$.

To check each of these as to whether it gives a saddle point or a local maximum or minimum, we use $D = f_{xx}f_{yy} - f_{xy}^2$. $f_{xx} = 12 - 12x$, $f_{yy} = 6$, and $f_{xy} = 6$. At $(0, 0)$, $D = 36 > 0$ so there is either a local maximum or a local minimum here. Since $f_x(0, 0) = 12 > 0$, it is a local minimum. The value of f at this point is $f(0, 0) = 0$. At $(1, -1)$, $D = -36 < 0$, so there is a saddle point. If we look at points along the x-axis, where $y = 0$, $f(x, 0) = 6x^2 - 2x^3$. (For very large positive values of x this becomes arbitrarily large and negative, while for large negative values of x it becomes large and positive. Hence there is no global maximum or minimum for the whole plane.)

Problem 5

Evaluate the line integral $\int_C (1 - x)ds$ where C is a portion of the circle of radius 2 and center $(0, 0)$, traversed from $(2, 0)$ to $(0, 2)$.

ANSWER: The curve can be parametrized as $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \frac{\pi}{2}$. The function $1 - x$ becomes $1 - 2\cos t$. $y' = 2\cos t$ and $x' = -2\sin t$. We can evaluate the integral as

$$\int_0^{\frac{\pi}{2}} (1 - 2\cos t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 2\int_0^{\frac{\pi}{2}} (1 - 2\cos t)\, dt = 2\left[t - 2\sin t\right]_0^{\frac{\pi}{2}} = \pi - 4.$$

Problem 6

What are the largest and smallest values that $f(x, y) = xy$ takes on, for (x, y) on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1?$$

At what points does f achieve those values?

ANSWER: We can view this as a constrained maximum/minimum (Lagrange multiplier) problem with the objective function $f(x, y) = xy$ and constraint $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$. Then $\overrightarrow{\nabla} f = y\vec{\imath} + x\vec{\jmath}$, and $\overrightarrow{\nabla} g = \frac{x}{4}\vec{\imath} + y\vec{\jmath}$. To find where these are parallel we solve $\overrightarrow{\nabla} f = \lambda(\overrightarrow{\nabla} g)$ for some value λ. We need to find x and y (and λ perhaps as a tool for finding x and y) that satisfy $y = \lambda\left(\frac{x}{4}\right)$, $x = \lambda\, y$, and $\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$.

From the first equation we get $\frac{y}{x} = \frac{\lambda}{4}$, while the second yields $\frac{y}{x} = \frac{1}{\lambda}$. (We divided by x and by λ: If $\lambda = 0$ then both x and y are 0, which does not fit $g(x, y) = 0$, so we can assume $\lambda \neq 0$. If $x = 0$ and $y = \lambda\left(\frac{x}{4}\right)$ with $\lambda \neq 0$, then $y = 0$, so again we have the point $(0, 0)$ which is not on the ellipse.) Setting these equal we have $\frac{\lambda}{4} = \frac{1}{\lambda}$ or $\lambda^2 = 4$. Hence $\lambda = \pm 2$.

If $\lambda = 2$, $x = \lambda\, y$ gives $x = 2y$. Using $g(x, y) = 0$ we have $\frac{4y^2}{8} + \frac{y^2}{2} = 1$, $y^2 = 1$, so $y = \pm 1$. If $y = 1$, $x = 2y = 2$, so we have the point $(2, 1)$. If $y = -1$ we get the point $(-2, -1)$.

If $\lambda = -2$, so $x = -2y$, we get in the same way the points $(-2, 1)$ and $(2, -1)$.

Now we have four points to consider, $(\pm 2, \pm 1)$. We evaluate the function $f(x, y) = xy$ at each of these. At $(2, 1)$ we get $f(x, y) = 2$. At $(2, -1)$ we get $f(x, y) = -2$. At $(-2, 1)$ we get $f(x, y) = -2$. At $(-2, -1)$ we get $f(x, y) = 2$. Hence the maximum value, $f(x, y) = 2$, occurs at the points $(2, 1)$ and $(-2, -1)$, and the minimum value, $f(x, y) = -2$, at $(2, -1)$ and $(-2, 1)$.

Problem 7

A thin plate covers the region in the plane bounded by the x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$, where $y \geq 0$.
The density of this plate is given by the function $\delta(x, y) = x + y$.

(a) Use an integral to evaluate the mass of this plate.

ANSWER: The plate is shown at the right. We compute the mass by integrating the density function through the region,

$$M = \int_0^1 \int_{y^2}^1 (x + y)\, dx\, dy = \int_0^1 \left(\frac{1}{2} + y - \frac{y^4}{2} - y^3 \right) dx = \frac{13}{20}$$

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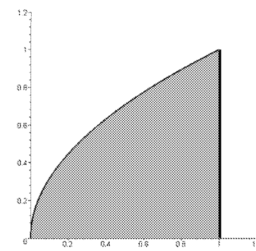


(b) Find the moment M_x of the plate about the x-axis.

ANSWER:

$$M_x = \int_0^1 \int_{y^2}^1 y(x + y)\, dx\, dy = \frac{3}{10}.$$

(c) Find the moment M_y of the plate about the y-axis.

ANSWER:

$$M_y = \int_0^1 \int_{y^2}^1 x(x + y)\, dx\, dy = \frac{19}{42}.$$

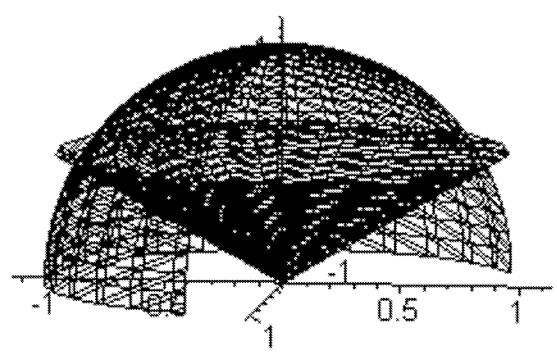
(d) Find the coordinates $(\overline{x}, \overline{y})$ of the center of mass of this plate.

ANSWER:

The x coordinate $\overline{x} = \frac{M_y}{M} = \frac{190}{273}$. The y coordinate $\overline{y} = \frac{M_x}{M} = \frac{6}{13}$.

Problem 8

Find the volume of the region that is above the cone $\phi = \frac{\pi}{3}$ and inside the sphere $\rho = 1$.
At the right is a figure cut away so that you can see inside the sphere.



ANSWER: We set up the integral in spherical coordinates. The radius ρ goes from 0 at the center out to 1 on the sphere. The horizontal direction θ goes all the way around from 0 to 2π. And the vertical angle ρ goes from 0 at the top, along the z-axis, down to $\frac{\pi}{3}$. Hence the volume integral is

$$\int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{3}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta = \frac{\pi}{3}.$$