The method of Undetermined Coefficients

Our text contains two techniques for finding a particular solution to a nonhomogeneous linear differential equation with constant coefficients in §17-2. The second one is called Variation of Parameters: This is a powerful tool and you should definitely learn to use Variation of Parameters if you intend to go on in subjects where such equations come up frequently. But it is also more complicated to apply in many situations. The other technique which works to solve some such problems is called Undetermined Coefficients, and when it does work it is generally much easier to use. In our class I will only expect you to solve problems where this technique can be used: You are welcome to use the more powerful Variation of Parameters technique on these problems, but I am guaranteeing that any nonhomogeneous linear differential equation with constant coefficients that you need to solve for this class will be one that can be done using Undetermined Coefficients.

Before proceeding, remember what the goal is for either Undetermined Coefficients or Variation of Parameters, in our somewhat restricted circumstances: We have learned how to find all solutions of an equation like $ay'' + by' + cy = f(x)$, or in theory higher order equations, which have (1) constant coefficients a, b, and c, on y and its derivatives, and (2) $f(x) = 0$ for "the other side", i.e. the equation is homogeneous. We want to handle the case where $f(x)$ is not zero, but as a step we still find all solutions to the homogeneous equation that results from replacing $f(x)$ by 0. The text calls this equation where $f(x)$ has been replaced by 0 the "complementary" equation. I will use y_h to denote the solutions to the (homogeneous) complementary equation. We have learned that if we can find any one solution to the real equation where $f(x)$ is not zero (I will use y_p to denote that particular solution) we can combine it with the general solution y_h to the complementary equation and get all solutions to the non-homogeneous equation we really want to solve. In summary, we find all solutions to $ay'' + by' + cy = 0$ and call that y_h, and any one solution to $ay'' + by' + cy = f(x)$ and call it y_p, and then $y_h + y_p$ will represent all solutions to the equation $ay'' + by' + cy = f(x)$ that we want to solve.

The problem we still have to solve is finding that particular solution y_p. Since any solution, any function that solves $ay'' + by' + cy = f(x)$, is OK, any way of finding y_p is fine including a lucky guess. In a "real world" problem you might well have an idea for a solution based on what you know about the problem. But since luck is not always with you, we need a more systematic way to find y_p, and that is what both Variation of Parameters and Undetermined Coefficients attempt to provide. You could describe Undetermined Coefficients as "guided guessing".

Here is how we carry out the guessing: If we are lucky and $f(x)$ is particularly simple, and it does not interact in bad ways with the solutions to the homogeneous equation, we may be able to find a solution y_p as follows. We will assume that $f(x)$ being simple means it is some combination of terms like e^{nx}, $\cos(kx)$, $\sin(kx)$, and polynomials $\alpha x^2 + \beta x + \gamma$. (If both cosine and sine terms are present, the same kx must appear in all of them.) We construct a candidate for y_p using the table on the next page.

How we use this is best shown by examples: As a simple first example, we will find the solutions of $y'' + 2y' - 3y = 6$. The auxiliary or characteristic equation is $r^2 + 2r - 3 = 0$, with roots $r = -3$ and $r = 1$. Hence we can write $y_h = C_1 e^{-3x} + C_2 e^{x}$. The right side of the equation in this case is just $f(x) = 6$, a polynomial of degree zero. Since 0 is not a root of the characteristic equation, we try for y_p a generic polynomial of degree zero, i.e. a constant: Let $y_p = F$ where F represents a constant yet to be determined. Then y_p' and y_p'' are both 0. Plugging this into the equation, for y_p to be a solution, we must have $0 + 0 - 3F = 6$. Hence the only possibility is $F = -2$, but that is good enough. The complete solution is obtained by adding y_h, $y(x) = C_1e^{-3x} + C_2e^x - 2$ as the general solution.

For a term in $f(x)$ which is a multiple of:	If:	Then use a term like: (A-F represent numbers to be determined)
$\sin(kx)$ or $\cos(kx)$	ki is not a root of the characteristic equation ki is a root of the characteristic equation	$A\cos(kx) + B\sin(kx)$ $Ax\cos(kx) + Bx\sin(kx)$
e^{nx}	n is not a root of the characteristic equation n is a single root of the characteristic equation n is a double root of the characteristic equation	Ce^{nx} $Cx\,e^{nx}$ $Cx^2\,e^{nx}$
A polynomial $\alpha x^2 + \beta x + \gamma$ of degree at most 2	0 is not a root of the characteristic equation	a polynomial $Dx^2 + Ex + F$ of the same degree as $\alpha x^2 + \beta x + \gamma$
	0 is a single root of the characteristic equation	a polynomial Dx^3+Ex^2+Fx of degree one more
	0 is a double root of the characteristic equation	a polynomial $Dx^4 + Ex^3 + Fx^2$ of degree two more

Another example: Solve $y'' - y' = 5e^x - \sin(2x)$. The characteristic equation is $r^2 - r = 0$, with roots $r = 0$ and $r = 1$. The complementary equation will thus have solutions $y_h = C_1e^{0x} + C_2e^{1x} = C_1 + C_2e^x$.

The terms in $f(x)$ are $5e^x$ and $-\sin(2x)$. Start with $5e^x$, which is a constant times e^{nx} where $n = 1$. Since 1 is a single root of the characteristic equation, we put the term $Cx\,e^x$ into y_p. Now $-\sin(2x)$ is a multiple of $\sin(kx)$ where $k = 2$, and 2 is not a root of the characteristic equation, so we include $A\cos(2x)+B\sin(2x)$ in our candidate solution. Thus we arrive at $y_p = A\cos(2x)+B\sin(2x)+Cx\,e^x$, and have to determine the coefficients A, B, and C. Then $y_p' = -2A\sin(2x) + 2B\cos(2x) + Ce^x + Cx\,e^x$, and $y_p'' = -4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x$. Putting these into the equation $y''-y' = 5e^x-\sin(2x)$ we get $-4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x+2A\sin(2x)-2B\cos(2x)-Ce^x-Cx\,e^x = 5e^x-\sin(2x)$. We collect together the terms from both sides with $\cos(2x)$ and get $-4A-2B = 0$. From the $\sin(2x)$ terms we get $-4B + 2A = -1$. The xe^x terms cancel out (a consequence of the fact that 1 was a root of the characteristic equation...) and the e^x terms give $C = 5$. Solving we get $A = -1/10$ and $B = 1/5$. Hence our particular solution is $y_p = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x + 5xe^x$. (You can check that y_p actually does work in $y'' - y' = 5e^x - \sin(2x)$.) Combining this with the general solution to the complementary equation y_h we get $y(x) = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x + 5xe^x + C_1 + C_2e^x$.

Note that this method does have limitations. As given here, for example, it would not apply to Problems 18 and 28 on section 17-16 in our text, since the limited table I included above does not suggest what to use for y_p when $f(x)$ includes the tangent function or has a product of two of the functions that are in the table. It would also not work for problem 29, if you were asked to find y_p, but in that problem you are given y_p. It will work for all the problems I will expect you to do.

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April 9, 2008