Series worksheet - answers **Math 222, Section 1** **Wilson** **February, 2004**

1. If $a_1 = 1$ and $a_2 = 3$ and $a_n = a_{n-1} - a_{n-2}$, what is a_5?
   We are given $a_1 = 1$ and $a_2 = 3$. Then $a_3 = a_2 - a_1 = 3 - 1 = 2$, so $a_4 = a_3 - a_2 = 2 - 3 = -1$, and $a_5 = (-1) - 2 = -3$.

2. Which of the following sequences $\{a_n\}$ converge? If a sequence does converge, what is its limit?

   (a) $a_n = \frac{n^2+3n-1}{4n^2-2}$.
   Each term is a ratio of polynomials of the same degree. Hence the sequence converges and the limit as the variable gets large will be the ratio of the highest-degree terms, $n^2/(4n^2) = \frac{1}{4}$.

   (b) $a_n = 1 + \sin\left(\frac{n\pi}{2}\right)$.
   As n runs through the positive integers, $\frac{n\pi}{2}$ goes through numbers which are the same as $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π, except for multiples of 2π. Hence $\sin\left(\frac{n\pi}{2}\right)$ cycles repetitively through the values 1, 0, -1, 0, and the sequence takes the values 2, 1, 0, 1 cycling forever. Hence there can be no number the values get close to and stay close to if "close to" means a distance of less than 2. Hence this sequence diverges.

   (c) $a_n = 1 + \sin n\pi$.
   An argument similar to (b) shows that the sequence values cycle through 1, 1, 1 forever. Hence this sequence converges to 1.

   (d) $a_n = 3 \times \left(\frac{1-(.7)^n}{1-.7}\right)$.
   Since $|.7| < 1$, $(.7)^n \to 0$ as $n \to \infty$. All other parts of a_n are constant, so $a_n \to 3 \times \left(\frac{1}{1-.7}\right) = \frac{3}{.3} = 10$. (You can also recognize the sequence as the partial sums of a geometric series with $a = 3$ and $r = .7$, coming to the same conclusion.)

3. Find a formula for the n^{th} partial sum of the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \ldots$. Then take the limit of that sum as $n \to \infty$ to evaluate the sum of the series.
   We had a formula for the n^{th} partial sum of a geometric series with initial term a and ratio r, $\frac{a(1-r^n)}{1-r}$. (Depending on how you number the terms, whether the first is numbered 0 or 1, your formula might have n replaced by $n + 1$.) Using $a = 1$ and $r = -\frac{1}{3}$ we have $\frac{(1-(-1/3)^n)}{1+1/3} = \frac{3}{4} - \frac{3}{4}\left(\frac{-1}{3}\right)^n$. Since $|\frac{-1}{3}| < 1$, the second term goes to zero and the sequence of partial sums converges to $\frac{3}{4}$.

4. Which of the following series converge? For each series give a reason for your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$
   This diverges. We can say that it is a p-series with $p = \frac{1}{2}$ which is less than 1. Or we could use the integral test. Or we could use comparison, each term (after the first) is bigger than the corresponding term in the harmonic series which we know diverges.

   (b) $\displaystyle\sum_{n=1}^{\infty} 2^{1/n}$
   The terms of this series are always at least $2^0 = 1$, so the terms don't go to zero. Hence this series must diverge.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n!}$
   This series has positive terms so we can use the ratio test. The ratio of successive terms $fraca_{n+1}a_n$ works out to $\frac{\sqrt{n+1}}{\sqrt{n}} \frac{n!}{(n+1)!}$. The first fraction has limit 1 as $n \to \infty$. The second simplifies to $\frac{1}{n+1}$ which has limit 0. Hence our limiting ratio $\rho = 0$ is less than 1 and the series converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos(n)}{n^2}$

This is a series whose terms are all at least zero, so we can use the comparison test. The numerator $1+\cos(n)$ is at most $1+1=2$, so the terms of the series are at most $\frac{2}{n^2}$. We know that series converges (twice a p-series with $p=2>1$) so the original series must converge.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{3^n}$

Since the numerator is the square of $\sin(n)$, it is non-negative, and hence the terms are all at least zero. Again we use the comparison test: The biggest $\sin^2(n)$ can be is $1^2=1$, so the n^th term is at most $\frac{1}{3^n}$. The series with those terms is a geometric series with $r=\frac{1}{3}<1$, so it converges, so the original series must also converge.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

We use the fact that for any positive integer n, $\ln(n)<n$. Thus the terms of this series are less than $\frac{n}{n^3}=\frac{1}{n^2}$ and that series converges (p-series with $p=2>1$.) Hence this series converges by comparison.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ \ln(n)}{2^n}$

We can use the ratio test (the terms are ≥ 0) and the ratio of successive terms is (after some algebraic rearrangement) $\frac{n+1}{n}\frac{\ln(n+1)}{\ln(n)}\frac{2^n}{2^{n+1}}$. The first two fractions each have limit 1, while the third is identically $\frac{1}{2}$. Hence $\rho=\frac{1}{2}<1$ so the series converges.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

The terms are ≥ 0 so we can use the ratio test. If we compute the ratio of successive terms we get $\frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}}$. The first fraction reduces to $n+1$. We can cancel that against part of the second denominator and have $\frac{n^n}{(n+1)^n}$. If we look at one over that we have $\frac{(n+1)^n}{n^n}=\left(\frac{n+1}{n}\right)^n$ $=\left(1+\frac{1}{n}\right)^n$. The limit of that as $n\to\infty$ is e. (This is a useful fact you may remember from earlier chapters, or you can get it using l'Hopital's rule.) Hence the limit of the ratios we had is one over e, which is less than 1, so the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Again we use the ratio test. The ratio of successive terms is $\frac{e^{n+1}}{e^n}\frac{n!}{(n+1)!}=\frac{e}{n+1}$, so the limit $\rho=0<1$ and the series converges.

5. For each of the following series tell whether it converges absolutely, conditionally, or not at all. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2+3}{1000n^2-500n+5}$

The limit of the terms of the series is $\frac{1}{1000}\neq 0$, so the series must diverge no matter whether the $\pm$ signs are included or not.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2+3}{1000n^3-500n+5}$

Now the limit of the terms is 0, and they are decreasing in size, so by the Alternating Series test (Leibniz' theorem) the series must converge. But if we take the absolute values of the terms, essentially just erasing the $(-1)^{n+1}$, we have a series whose terms are bigger than those of $\frac{1}{1000}\times$ the harmonic series and so it diverges. Thus this series converges conditionally.

(c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+3}{1000n^4-500n+5}$
This time the terms are still going to zero, in a decreasing fashion, so the series still converges by the Alternating Series test. But this time the terms, for large values of n, "look like" $\frac{1}{n}^2$ and so we expect the series of absolute values to converge. It does: The best argument is to use the Limit Comparison test comparing it to $\sum 1/n^2$, where the ratio of a term from our series and a term from $1/n^2$ has limit $\frac{1}{1000}$ or 1000 depending on which term you put on top. Since this limit exists as a finite, non-zero, number, this series must converge or diverge exactly as $\sum \frac{1}{n^2}$ does, and that is a convergent p-series. Hence the series of absolute values does converge, so the original series converges absolutely.

(d) $\sum_{n=1}^{\infty}\frac{1}{(-5)^n}$
If we start with the absolute values we get $\sum\left(\frac{1}{5}\right)^n$ which is a geometric series with $r=\frac{1}{5}$, less than 1 in absolute value. Hence the series of absolute values converges, so the series itself converges absolutely. Once we know that we don't need to check for convergence, since absolute convergence implies convergence.

(e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$, for $p>1$.
The series of absolute values is $\sum\frac{1}{n^p}$, a p-series, with $p>1$ given to us. Hence it converges and the original series converges absolutely.

(f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$, for $0<p\leq 1$.
This time the series of absolute values will be a p-series with $p\leq 1$, which does not converge. Hence the series is either conditionally convergent or not convergent at all. But for $p>0$, $n^p\to\infty$ as $n\to\infty$, so the terms of the series decrease to zero. Hence the series does converge, by the Alternating Series test. Thus the series is conditionally convergent.

(g) $\sum_{n=1}^{\infty}\frac{\cos n\pi}{\sqrt{n}}$
Evaluating $\cos n\pi$ at $n=1,\ 2,\ 3,\ldots$ we get successively $-1,\ 1,\ -1,\ldots$ repitively. Hence this is an alternating series. The size of the terms, $\frac{1}{\sqrt{n}}$, decreases with limit zero. Hence the series does converge, by the Alternating Series test. But the series of absolute values, essentially the same series as in problem 4(a), diverges. So the series converges conditionally.

(h) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n!}$
The series of absolute values has as a typical term $\frac{\ln(n)}{n!}$ which is positive for $n>1$. Applying the ratio test we find the ratio of successive terms to be $\frac{\ln(n+1)}{\ln(n}\frac{n!}{(n+1)!}$. The first fraction has limit 1 while the second is equal to $\frac{1}{n+1}$ which has limit 0. Hence the limiting ration $\rho=0<1$ and the series of absolute values converges. That shows the series itself converges absolutely.

6. The series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots$ (where the denominators run through the odd numbers) converges to $\frac{\pi}{4}$.

(a) Although I already said it converges, prove that using some test we have had.
This is an alternating series with terms decreasing to zero, so it converges by the Alternating Series test.

(b) If we wanted to approximate π we could use the first n terms of that series, for some n, and multiply the sum by 4. What terms should we use if we need to get our value for π correct to within ± 0.01?
If we want to get $4\times$ the sum approximated to within ± 0.01, we had better reduce the error

in approximating the sum to within $\pm\frac{0.01}{4} = \pm\frac{1}{400}$. Since we are using an alternating series, we can use the terms such that the first one we omit has size less than $\frac{1}{400}$. Thus we want an odd number N such that $\frac{1}{N} < \frac{1}{400}$. If we solve $\frac{1}{N} = \frac{1}{400}$ we get $N = 400$. Since only odd denominators appear in the series, if we stop with $\pm\frac{1}{399}$ the first omitted term would be $< \frac{1}{400}$. So we can use $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{399}$. If you look at the terms you see that the plus signs go where the denominator is one more than a multiple of four, and minus signs where the denominator is one less than a multiple of four, so the correct sign on that last term is $-$. (If we calculate the sum of those terms we get $0.7841481712\ldots$, and if we multiply that by 4 we get $3.136592685\ldots$, which is just within ± 0.01 of $\pi \approx 3.1415926535\ldots$)