Approximation worksheet **Math 222, Section 1** **Wilson** **March, 2004**

Solutions

1. Calculate e with an error of at most 10^{-7}.

   I will use part of the Maclaurin series $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \cdots$ with $x = 1$. We need to determine how many terms we need to include in order to meet the accuracy requirement. We use the remainder term $R_n(x) = \frac{f^{(n+1)(c)}}{(n+1)!}x^{n+1}$ that corresponds to $a = 0$, since this is a Maclaurin series. We use $x = 1$, and $0 < c < 1$, to estimate how big $R_n(x)$ might be. The derivatives of e^x are all just e^x, so $f^{(n+1)}(c) = e^c$. The largest e^c can be for $0 < c < 1$ is $e^1 = e < 3$, since e^x is an increasing function. $x^{n+1} = 1^{n+1} = 1$, so $|R_n(x)| \leq \frac{3}{(n+1)!} \times 1$. We want $|R_n(x)| \leq 10^{-7}$ so we make $|R_n(x)| \leq \frac{3}{(n+1)!} < 10^{-7}$ and solve for n. We get $(n + 1)! > \frac{3}{10^{-7}} = 3 \times 10^7 = 30,000,000$. Trying values of n we get eventually that $10! = 3,628,800$ and $11! = 39,916,800$ so the first value of n satisfying $(n + 1)! > 30,000,000$ is $n = 10$. Hence we know that if we use the terms $1 + x + \cdots + \frac{x^10}{10!}$, with $x = 1$, we should have sufficient accuracy. That gives us $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{10!} \approx 2.718281801$. The actual value of e through that many places is 2.718281828, and these do agree to within ± 0.0000001 as required. If we had used only the terms through $\frac{1}{9!}$ we would get 2.718281526, which does not meet the required accuracy.

2. If we use $x - \frac{x^3}{3!}$ to approximate $\sin(x)$, for what values of x is the approximation correct to within ± 0.0003?

   I will do this one in two ways. First, using the remainder term $R_n(x)$: The terms given are the first two non-zero terms of the Maclaurin series for $\sin(x)$. But since the even-power terms are all zero, this is in fact the same as the terms through the 4^{th} power of the series. Hence we can get our most accurate results if we think of this as approximation by the terms $0 + x + 0x^2 + \frac{x^3}{3!} + 0x^4$, i.e. $n = 4$. The remainder term $R_4(x)$ requires the 5^{th} derivative of $\sin(x)$, which is $\cos(x)$. We need to find what x values make $|\frac{\cos(c)}{5!}x^5| \leq 0.0003$ for every c between 0 and x. We can't even be sure what c is when we know what x is, so certainly we don't know c here. But we can say that $|\cos(c)| \leq 1$ since that is true for any c. Hence we want to know what x values make $|\frac{\cos(c)}{5!}x^5| \leq |\frac{1}{120}x^5| \leq 0.0003$. We can solve that last inequality to get $|x^5| \leq 120 \times 0.0003 = 0.036$, or (since $|x^5|$ increases as we move away from 0) $|x| \leq \sqrt[5]{0.036} \approx 0.514352$. Thus if $-0.514352 \leq x \leq 0.514352$ the approximation will be within the required tolerance. (If you use a calculator or computer to plot $(x - \frac{x^3}{6} - \sin(x)$ for $-0.6 < x < 0.6$, you will see this works!)

   But in this case we have another method available. The Maclaurin series for $\sin(x)$, $x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, is an alternating series for any (non-zero) value of x: If $x > 0$ the terms go +, −, +, etc., while for $x < 0$ they reverse and go −, +, −, etc. but in either case the signs alternate. So we can use the alternating series error estimation process instead of Taylor's theorem. This says the error will be (in size) at most the size of the first omitted term. If we ignore the zero terms, the first omitted term is $\frac{x^5}{5!} = \frac{x^5}{120}$. So we find what values of x make $|\frac{x^5}{120}| \leq 0.0003$. As you can see in the previous paragraph, that is exactly what we solved to find the usable x values in the other approach, so we get the same conclusion! (The moral is that if the series alternates you may be able to save some work...)

3. If $\cos(x)$ is replaced by $1 - \frac{x^2}{2}$, for $|x| < 0.5$, estimate the error resulting. Does this approximation tend to be larger or smaller than the actual value of $\cos(x)$ for these x values?

   We think of this as $1 + 0x - \frac{x^2}{2!} + 0x^3$, i.e. the terms of the Maclaurin series for $\cos(x)$ through the third degree term. We could use the Taylor remainder term $R_3(x)$, but this time we note the series is an alternating series and use that to reduce the work. The first omitted term is $\frac{x^4}{24}$, so we know the error is at most whatever is the biggest $\frac{x^4}{24}$ can be for $-0.5 < x < 0.5$. Since x^4 increases (in absolute value or "size") as we move away from 0, the largest value will be at $x = -0.5$ or at $x = 0.5$: in fact they give the same result, $\frac{(0.5)^4}{24} = \frac{1}{384} \approx 0.0026$. Hence we can say the error will be at most

0.0026. The alternating series test also said the sum of the terms we use will be too small if the first omitted term is positive, and too large if the first omitted term is negative. Since $\frac{x^4}{24}$ is certainly positive except at $x = 0$, the approximation will be smaller than $\cos(x)$ except at $x = 0$ where it is exactly right.

4. The approximation $\sin(x) \approx x$ is sometimes used for small values of x. How good is it, if we only use it for $|x| < 0.001$? For which of those values of x will x be less than $\sin(x)$? greater?

   x is the first (non-zero) term in the Maclaurin series for $\sin(x)$, $0+x+0x^2+\cdots$. We could use either the Taylor remainder term, with $n = 2$ since this is really the series through the 2^{nd} degree term, or we can view the series as an alternating series. Since the latter is simpler I will use that: The error will be the magnitude of the first omitted term, $\left|\frac{x^3}{6}\right|$. For $|x| < 0.001$, that is at most $\frac{(0.001)^3}{6} = 1.6666\ldots\times 10^{-10}$, so the approximation is within that error bound for all the x's involved. The first omitted term, $-\frac{x^3}{6}$, is negative when $x > 0$ and positive when $x < 0$. Hence the approximation is too big (i.e. $x > \sin(x)$) when $x > 0$ and too small ($x < \sin(x)$) when $x < 0$.

5. (a) Use the remainder term in Taylor's theorem to estimate the error that results if we replace e^x by $1 + x + \frac{x^2}{2!}$ for $|x| < 0.1$.

   The polynomial $1 + x + \frac{x^2}{2!}$ is the terms through the 2^{nd} degree term of the Maclaurin series for e^x. The remainder term $R_2(x)$ will involve the 3^{rd} derivative of e^x which is still e^x, so we have $R_2(x) = \frac{e^c}{3!}x^3$. For the x values involved, using the fact that c is between 0 and x, e^c is at most $e^{0.1}$. Now in fact $e^{0.1}$ is about 1.105, but it would be hard to justify that if we could not compute e^x and if we could compute e^x this problem would be senseless! So I will use the much looser bound that $e^{0.1} < e^1 < 3$. The $|x^3|$ part of $|R_2(x)|$ will be at most $(0.1)^3 = 0.001$, so we have $|R_2(x)| < \frac{3\times 0.001}{6} = 0.0005$. Hence the error in replacing e^x by $1 + x + \frac{x}{2}$, for $|x| < 0.1$, is at most 0.0005.

   (b) For $x < 0$, the series $1+x+\frac{x^2}{2!}+\ldots$ is an alternating series. Use the Alternating Series estimation process to estimate the error that results if we replace e^x by $1 + x + \frac{x^2}{2!}$ for $-0.1 < x < 0$.

   The first term omitted is $\frac{x^3}{6}$ so the error is at most $\left|\frac{x^3}{6}\right|$, which will be largest (among the x values involved) at $x = -0.1$. Hence the error is at most $\frac{0.001}{6} = 0.0001666\ldots$.

   Compare your answers to (a) and (b): How can they be different and both be correct?

   The estimate of the error we got in (b) is only $\frac{1}{3}$ as big as the estimate in (a). There are two differences between what we are doing in (a) and (b). For one thing we are solving a different problem: Over a more restricted range of x values we might well be able to say that the error would be smaller. In addition we used a different method to solve the problem, alternating series vs. Taylor's theorem remainder term. Since each method just says "the error is at most such-and-such", they can give different answers and both be correct. I might also say in this case the error is at most 1: That happens to be true, since both methods got much smaller bounds, but it is not as useful.

6. (Continuing from our quiz of 3/5/04...) The first two terms of the Taylor series for $\sqrt{x}$ at $a = 1$ are $1 + \frac{1}{2}(x - 1) = \frac{1}{2} + \frac{x}{2}$. If we approximate $\sqrt{0.95}$ using that polynomial, how accurate will the results be? Use one of the theorems we have had, and then compare the answers you get from the polynomial and $\sqrt{0.95}$ using a calculator. Are these consistent with the theoretical results?

   Using $x = 0.95$ in the polynomial we get $1 + \frac{1}{2}(0.95 - 1) = 1 - 0.025 = 0.975$ as our approximate value for $\sqrt{0.95}$. The series is not alternating (at least through these terms) so we have to use the remainder term $R_1(x)$ (1 since this polynomial took us through the 1^{st} degree term of the series) to estimate the error. We have $R_1(x) = \frac{f''(c)}{2}(x-1)^2$, where $f(x) = \sqrt{x}$, $x = 0.95$ and c is between 1 and x. The second derivative of $f(x)$ is $-\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4(x)^{\frac{3}{2}}}$. The power in the denominator is an increasing function on $0.95 < c < 1$, so the fraction will be largest (in absolute value) when the denominator is smallest, i.e. $|f''(c)| = |-\frac{1}{4c^{\frac{3}{2}}}| < \frac{1}{4(0.95)^{\frac{3}{2}}}$. Thus $|R_1(0.95)| \le \frac{1}{8\times(0.95)^{\frac{3}{2}}}(0.05)^2 = \frac{0.0025}{8\times(0.95)^{\frac{3}{2}}} = \frac{0.0003125}{(\sqrt{0.95})^3}$. But if we had to use a polynomial to approximate $\sqrt{0.95}$, surely we don't know $(\sqrt{0.95})^3$! Since we want to be able to say that $\frac{1}{(\sqrt{0.95})^3}$ is less than something, we want to find a number M such that $\frac{1}{(\sqrt{0.95})^3} < M$, i.e. $(\sqrt{0.95})^3 > \frac{1}{M}$. We want M as small as we can conveniently make it: E.g. $(\frac{1}{\sqrt{0.95})^3}$ is surely less than 1000 but that would hardly be useful. The nearest "obvious" number is 1, but it is not true that $\frac{1}{(\sqrt{0.95}} < 1$ so we can't use that. To construct a number we can easily find the $\frac{3}{2}$ power of, start with something that is a nice square. If we pick $0.81 = (0.9)^2$, we get the $\frac{3}{2}$ power of 0.81 will be $(0.9)^3 = 0.729$. So now we have (a) $(0.81)^{\frac{3}{2}} = 0.729$ and (b) $(0.81)^{\frac{3}{2}} < (0.95)^{\frac{3}{2}}$ so $\frac{1}{(\sqrt{0.95})^3} < \frac{1}{(\sqrt{0.81})^3} = \frac{1}{0.729} \approx 1.3717$. Now we go back to the remainder term and have the error bounded by $(0.0003125) \times (1.3717) \approx 0.0004287$. Hence the difference between $\sqrt{0.95}$ and our approximation 0.975 should be at most 0.0004287. Using a calculator I get $\sqrt{0.95} \approx 0.97468$, so the difference is actually about 0.00032, which is indeed less than 0.0004287.

7. How many terms of the Maclaurin series for $\ln(x+1)$ should you include to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}?

   First, what is the series? You can take derivatives and find the coefficients that way, or do as we did with one of our very first examples of power series: Note that $\ln(1+x) = \int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 - x^4 + \cdots)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$. (I dropped the absolute value signs, we will be applying this only where $1 + x > 0$. In evaluating $\ln(1.1)$ using $\ln(1 + x)$ we must be using $x = 0.1$, and at that value of x this is an alternating series. (You could also do this with the remainder term.) Using alternating series error estimation we use the terms through the point where the first term omitted is in magnitude less than 10^{-8}. The terms in general are $\pm\frac{x^n}{n} = \pm\frac{(0.1)^n}{n}$. Hence we want to find n such that $\frac{(0.1)^n}{n} < 10^{-8}$. Note that this amounts to $\frac{10^{-n}}{n} < 10^{-8}$: If we choose $n = 8$ that must be true but conceivably it is true for smaller smaller n. If we try $n = 7$ we get $\frac{10^{-7}}{7} \approx 1.42857 \times 10^{-8}$ which is <u>not</u> smaller than 10^{-8}, so we use enough of the series that $\pm\frac{(0.1)^8}{8}$ is the first term omitted. Thus we compute $\ln(1.1)$ as approximately $(0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \cdots + \frac{(0.1)^8}{8}$ which works out to about .09531018095. (And a calculator gives $\ln(1.1)$ as about .09531017980, for comparison, so they certainly do agree to the precision requested.)

8. Use the Alternating Series test to decide how many terms of the Maclaurin series for $\arctan(x)$ $(\tan^{-1}(x))$ should be included to get $\frac{\pi}{4}$ within 10^{-3}.

   Again you can find the arctangent series either by using the formula for coefficients in a Maclaurin series or by working from the geometric series $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ and integrating to get $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. To use this to get $\frac{\pi}{4}$ we evaluate $\tan^{-1}(1) = \frac{\pi}{4}$, so we want to use $x = 1$ in enough terms of the series to make the error less than 0.001. For any x other than zero the even powers would all be positive so with the $\pm$ signs included this is an alternating series. Hence we use enough terms that the first one omitted is less than 0.001 in size, i.e. we go up to an odd number n such that $\frac{(1)^n}{n} = \frac{1}{n}$ is less than $0.001 = \frac{1}{1000}$. That happens when $n = 1001$, so we use the terms through $\pm\frac{1}{999}$. (Since the terms where the denominator is one less than a multiple of 4 get minus signs, that is more specifically $-\frac{1}{999}$. This is as much as you were asked to do. If I ask a computer to add up those terms, $1 - \frac{1}{3} + \frac{1}{5} + \cdots - \frac{1}{999}$, I get 0.7848981639, and the difference between that and $\frac{\pi}{4}$ is about 0.0004999996 which is indeed less than 10^{-3}.