Here is a worked out solution to Problem 21 on Page 576. I was asked about this in the problem session this morning (4/27/04) and I promised to get back to it, but there is enough algebra to work out that I decided it would be good to type it up nicely...

We are given two vectors $\vec{a}$ and $\vec{b}$ that are not collinear, i.e. are not in the same direction, and a third vector $\vec{r}$ in the same plane as $\vec{a}$ and $\vec{b}$. (This is a strange wording, since in section 13.3 everything is in two dimensions so it is hard <u>not</u> to be in the same plane. There is a generalized version of this true in 3-space where the assumption makes sense...) We are asked to find numbers k and m such that $\vec{r}$ can be represented as the combination $k\vec{a}+m\vec{b}$ of $\vec{a}$ and $\vec{b}$.

We can represent each of the vectors we start with as a combination of the basis vectors $\vec{\imath}$ and $\vec{\jmath}$: $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath}$ and $\vec{r} = r_1\vec{\imath} + r_2\vec{\jmath}$ for some numbers a_1, a_2, etc.

Now assume that some numbers k and m exist doing what we want. Then $\vec{r} = k\vec{a} + m\vec{b}$. Putting in the above representations of $\vec{r}$, $\vec{a}$, and $\vec{b}$, we have

$$r_1\vec{\imath} + r_2\vec{\jmath} = k(a_1\vec{\imath} + a_2\vec{\jmath}) + m(b_1\vec{\imath} + b_2\vec{\jmath}) = (ka_1 + mb_1)\vec{\imath} + (ka_2 + mb_2)\vec{\jmath}.$$

We can pick out the $\vec{\imath}$ pieces from both sides, which must be equal, and similarly the $\vec{\jmath}$ pieces, to get

$$r_1 = ka_1 + mb_1 \qquad \text{and} \qquad r_2 = ka_2 + mb_2.$$

In those equations we know the r's, the a's, and the b's, so we have two equations to solve for the two unknown quantities k and m. There are various ways to solve them. Here is one. Multiply the first equation by a_2 and the second by a_1, getting

$$r_1a_2 = ka_1a_2 + ma_2b_1 \qquad \text{and} \qquad r_2a_1 = ka_1a_2 + ma_1b_2.$$

Now subtract the first of those equations from the second, to get

$$r_2a_1 - r_1a_2 = m(a_1b_2 - a_2b_1).$$

Next divide by $(a_1b_2 - a_2b_1)$ and we have

$$m = \frac{r_2a_1 - r_1a_2}{a_1b_2 - a_2b_1}.$$

(Note that this division is OK: If $a_1b_2 - a_2b_1$ were zero, then $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ so the two vectors would point in the same direction.)

We can solve instead for k: Rather than multiplying the first equation by a_2 and the second by a_1, use b_2 and b_1. The rest of the work goes the same way, to get

$$k = \frac{r_1b_2 - r_2b_1}{a_1b_2 - a_2b_1}.$$

Now all of this assumed we could find such numbers k and m, but each step is in fact reversible. So if we use the numbers we just found, then $\vec{r} = k\vec{a} + m\vec{b}$.