First Midterm Exam February 24, 2004 Answers

Problem 1 Evaluate the integrals:

(a) $$\int \sin^2(x)\,\cos^3(x)\,dx$$

Answer:
This product of an even power and an odd power of sines and cosines we can handle by putting one copy of $\cos(x)$ with dx and using $\cos^2 x = 1 - \sin^2 x$ on the rest of $\cos^3(x)$. We get $\int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$ $= \int(\sin^2(x) - \sin^4(x))\cos(x)\,dx$. If we let $u = \sin(x)$ so that $du = \cos(x)\,dx$, this is $\int(u^2 - u^4)\,du =$ $\frac{1}{3}u^3 - \frac{1}{5}u^5 + C$, or substituting the original variable $\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$.

(b) $$\int_0^{\frac{\pi}{6}} \cos^2(x)\sin^2(x)\,dx$$

Answer:
I'll put off worrying about the limits of integration until we complete the antidifferentiation. This time both powers are even so we use $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$. We get $\frac{1}{4}\int(1+\cos(2x))(1-\cos(2x))\,dx = \frac{1}{4}\int(1-\cos^2(2x))\,dx$. Now we use one of those identities again to get $\frac{1}{4}\int(1-\frac{1}{2}(1+\cos(4x)))\,dx$ $= \frac{1}{8}\int(1-\cos(4x))\,dx = \frac{1}{8}(x - \frac{1}{4}\sin(4x)) + C = \frac{x}{8} - \frac{1}{32}\sin(4x) + C$. Now going back to the definite integral we need to evaluate that at $x = \frac{\pi}{6}$ and $x = 0$ and subtract, getting $\frac{\pi}{48} - \frac{1}{32}\sin(\frac{2\pi}{3}) = \frac{\pi}{48} - \frac{\sqrt{3}}{64}$.

Problem 2

(a) Evaluate the integral: $$\int \frac{4x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$$

Answer:
I would draw a right triangle and label it: One of the acute angles I will label θ, the hypoteneuse 1, the leg opposite to θ I will label x, and then if the leg adjacent to θ is labelled $\sqrt{1-x^2}$ the labelling is consistent with the Pythagorean theorem. We have $x = \sin\theta$ so $dx = \cos(\theta)d\theta$, and $\sqrt{1-x^2} = \cos(\theta$. Putting those in the integral we get $4\int \frac{4\sin^2\theta}{\cos^3\theta}\cos(\theta)d\theta = 4\int \frac{\sin^2\theta}{\cos^2\theta}d\theta = 4\int \tan^2\theta\,d\theta$. Now use the fact that $\tan^2\theta = \sec^2\theta - 1$ and we have $4\int(\sec^2\theta - 1)d\theta = 4(\tan\theta - \theta) + C$. From the triangle we have $\tan\theta = \frac{x}{\sqrt{1-x^2}}$ and $\theta = \arcsin(x)$, so we can write the answer as $\frac{4x}{\sqrt{1-x^2}} - 4\arcsin(x) + C$.

(b) Convert this integral to an integral of a trigonometric function: You do not have to evaluate the resulting integral.

$$\int \frac{dx}{\sqrt{4+x^2}}$$

Answer:
Again I will draw a triangle. I label the leg adjacent to θ as 2, the one opposite as x, and the hypoteneuse as $\sqrt{1+x^2}$. Then $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$, and $\sqrt{4+x^2} = 2\sec\theta$. Hence the integral becomes $\int \frac{2\sec^2\theta}{2\sec\theta}d\theta = \int \sec\theta\,d\theta$.
This is all you were required to do, but we could evaluate it as $\ln|\tan\theta + \sec\theta| + C$ and substitute back for x to get $\ln|\frac{x}{2} + \frac{\sqrt{4+x^2}}{2}| + C$.

Problem 3 Evaluate the integrals:

(a) $$\int e^x\,(x^2 - 5x)\,dx$$

Answer:
This product suggests integration by parts. If we let $dv = (x^2-5x)dx$ then v will be cubic, the wrong way to go. So we let $u = x^2 - 5x$ and $dv = e^x\,dx$. Then $du = (2x-5)dx$, $v = e^x$, and the integral becomes

$e^x(x^2-5x)-\int(2x-5)e^x\,dx$. This requires one more integration by parts. Let $u=2x-5$ so $du=2\,dx$, and $dv=e^x\,dx$ so $v=e^x$. Now we get $e^x(x^2-5x)-e^x(2x-5)+\int e^x\,2\,dx = e^x(x^2-5x)-e^x(2x-5)+2e^x+C$. We can simplify this a little as $e^x(x^2-7x+7)+C$.

(b) $\displaystyle\int \arctan(x)\,dx$

Answer:
Use integration by parts. Let $u=\arctan(x)$ and $dv=dx$, so $du=\frac{1}{1+x^2}dx$ and $v=x$. Then the integral becomes $x\arctan(x)-\int\frac{x\,dx}{1+x^2}$. Now we use a simple substitution: Let $u=1+x^2$, so $du=2x\,dx$, and that last integral gives $\frac{1}{2}\ln|1+x^2|+C$, so the answer is $x\arctan(x)-\frac{1}{2}\ln|1+x^2|+C$.
(We can drop the absolute value signs this time if we note that $1+x^2$ is never negative.)

Problem 4 Evaluate the integral:

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{5}}}$$

Answer:
Integrating $(x-1)^{-\frac{2}{5}}$ is pretty easy, but we must note that the denominator is zero when $x=1$, which is within the interval we are integrating over. So we will need to write this using limits. I will first work out the indefinite integral: $\int(x-1)^{-\frac{2}{5}}\,dx=\frac{5}{3}(x-1)^{\frac{3}{5}}+C$.
Now we need to evaluate this on the interval $[0,b]$, as $b\to 1^-$, and on $[a,3]$, as $a\to 3^+$, and add. The first gives $\frac{5}{3}(b-1)^{\frac{3}{5}}-\frac{5}{3}(-1)^{\frac{3}{5}}$ which has limit $-\frac{5}{3}(-1)^{\frac{3}{5}}=\frac{5}{3}$ as $b\to 1^-$. The second gives $\frac{5}{3}(2)^{\frac{3}{5}}-\frac{5}{3}(a-1)^{\frac{3}{5}}$ which goes to $\frac{5}{3}(2)^{\frac{3}{5}}$ as $a\to 1^+$. Hence the answer is $\frac{5}{3}(2^{\frac{3}{5}}+1)\approx 4.192860946$.

Problem 5 Evaluate the integrals:

(a) $\displaystyle\int_1^\infty x\,e^{-x}\,dx$

Answer:
This is an improper integral we evaluate as $\displaystyle\lim_{b\to\infty}\int_1^b x\,e^{-x}\,dx$. Using parts, with $u=x$ and $dv=e^{-x}\,dx$ so $du=dx$ and $v=-e^{-x}$, we have $\displaystyle\lim_{b\to\infty}\left[\left[-xe^{-x}\right]_1^b+\int_1^b e^{-x}\,dx\right]=\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^b=\lim_{b\to\infty}\left[-b\,e^{-b}-e^{-b}\right]+\frac{2}{e}$. Now $\displaystyle\lim_{b\to\infty}e^{-b}$ is pretty clearly 0, but the limit of $b\,e^{-b}$ is not so obvious. Think of that as $\frac{b}{e^b}$ and note that both numerator and denominator go to ∞. With that justification apply l'Hopital's rule: The derivatives of numerator and denominator give $\frac{1}{e^b}$ which clearly goes to 0. So the answer is $\frac{2}{e}$.

(b) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+2}}\,dx$

Answer:
Again I will put off the limits and first work out $\int\frac{1}{\sqrt{x+2}}\,dx$. This is just a power-rule integral, giving $2\sqrt{x+2}+C$. Now set up the integral as it really should be, $\displaystyle\int_0^\infty\frac{1}{\sqrt{x+2}}dx=\lim_{b\to\infty}\int_0^b\frac{1}{\sqrt{x+2}}dx=\lim_{b\to\infty}\left[2\sqrt{x+2}\right]_0^b=\lim_{b\to\infty}2\sqrt{b+2}-2\sqrt{2}$. As $b\to\infty$, $\sqrt{b+2}$ increases without bound, so this limit does not exist, and the integral diverges.

Problem 6 For each series, tell whether it converges or diverges and give a reason for your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3n+2}{2n^2-1}$

Answer:
As $n \to \infty$, the terms of this series approach $\frac{1}{2}$, not 0, so by the n^{th} term test this series must diverge.

(b) $\displaystyle\sum_{n=2}^{\infty} e^{-n}$

Answer:
The terms of this series do go to zero, so it might converge. If we try the ratio test we have

$$\rho = \lim_{n\to\infty} \frac{e^{-(n+1)}}{e^{-n}} = \lim_{n\to\infty} \frac{1}{e} = \frac{1}{e} < 1,$$

so the series converges using the ratio test. It is also pretty easy to work out $\int_2^\infty e^{-x}\,dx$ and find that it converges, so the integral test also shows this series converges. And for a third version, this is really a geometric series, each term is $\frac{1}{e}$ times the previous one so the ratio in the geometric series is $\frac{1}{e}$ which is between ± 1 and so again we know the series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 1000}{6 + 7^n}$

Answer:
This series does converge. You could show this using the limit comparison test, but here is another way: Consider separately the series $\sum \frac{3^n}{6+7^n}$ and $\sum \frac{1}{6+7^n}$. The first of these is term-by-term less than $\sum \frac{3^n}{7^n}$ $= \sum \left(\frac{3}{7}\right)^n$ which is a converging geometric series, $-1 < r = \frac{3}{7} < 1$. The second is term-by-term less than $\sum \frac{1}{7^n}$ and that is similarly a converging geometric series. Hence the series $\sum \frac{3^n}{6+7^n}$ and $\sum \frac{1}{6+7^n}$ both converge by the comparison test. But the series in the problem is the first series plus 1000 times the second, and so it converges by our theorem on doing arithmetic with series.

Problem 7 For each series, tell whether it converges absolutely, converges conditionally, or does not converge at all, and give a reason for your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(-4)^n}$

Answer:
If we start by looking at the series of absolute values, we have $\sum \frac{1}{4^n}$ which is a geometric series with ratio between ± 1 so it converges. Hence this series converges absolutely and we don't need to check anything further.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{100n}\right)$

Answer:
The terms of this series have limit $\frac{1}{100} \neq 0$, so this series must not converge at all.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

Answer:
The series of absolute values $\sum \frac{\ln}{n}$ is term-by-term (for $n > 3$) greater than the terms of the harmonic series which diverges. So the series of absolute values does not converge. But the terms $\frac{\ln n}{n}$ do decrease (for $n > 3$) with limit zero. (you can check this using calculus, the derivative is negative so the values are decreasing and l'Hopital's rule shows the limit is zero, but I am willing to accept this without proof.) So with the alternating signs included the series does converge, by the Alternating Series test (Leibniz' theorem). Hence the series is conditionally convergent.

Problem 8

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

Hints:

(i) Find numbers A and B such that $\frac{2}{n^2+4n+3} = \frac{A}{n+1} + \frac{B}{n+3}$.

(ii) Find an expression for the general n^{th} partial sum of the series, making use of the A and B you found in (i).

(iii) Use the definition of the sum of an infinite series.

Answer:
Following the recipe above...

(i) Assume $\frac{2}{n^2+4n+3} = \frac{A}{n+1} + \frac{B}{n+3}$. Multiply both sides by n^2+4n+3 which is the same as $(n+1)(n+3)$. This gives $2 = A(n+3) + B(n+1) = (A+B)n + (3A+B)$. Now the constants on the two sides must be equal, so $3A+B = 2$. And the multipliers on n must also be equal: There is no n term on the left, i.e. the term is $0n$, so $A+B = 0$. That makes $B = -A$. Substitute that into $3A+B = 2$ and get $3A - A = 2$ or $2A = 2$, so $A = 1$. Then $B = -A = -1$. Thus we have $\frac{2}{n^2+4n+3} = \frac{1}{n+1} + \frac{-1}{n+3}$ which you can check is true by adding the fractions.

(ii) Write down the first few terms of the series, in the form $\frac{1}{n+1} - \frac{1}{n+3}$:

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) \cdots$$

Notice that the second term in each set of parentheses cancels against the first term in the parentheses two steps later. Thus the sum of the first n terms, for n at least 3,

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

has every term cancelling except $\frac{1}{2}$, $\frac{1}{3}$, and two terms at the end, $-\frac{1}{n+2}$ and $-\frac{1}{n+3}$, and the partial sum is $\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$.

(iii) The sum of the series is by definition the limit of the sequence of partial sums. As $n \to \infty$, the last two terms in each partial sum are getting smaller with limit 0. Hence the sum of the series is $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.