The method of Undetermined Coefficients

Our text contains, in section 18-10, one technique for finding a particular solution to a nonhomogeneous linear differential equation with constant coefficients, called Variation of Parameters. This is a powerful tool and you should definitely learn to use Variation of Parameters if you intend to go on in subjects where such equations come up frequently.

There is another technique which works to solve some such problems, called Undetermined Coefficients, which when it does work is generally much easier to use. In Math 222 I will only expect you to solve problems where this technique can be used: You are welcome to use the more powerful Variation of Parameters technique on these problems, but I am guaranteeing that any nonhomogeneous linear differential equation with constant coefficients that you need to solve for this class will be one that can be done using Undetermined Coefficients. The following introduction to the technique is adapted from a later edition of our text, the 8^{th} edition instead of the 5^{th}.

Before proceeding, remember what the goal is for either Undetermined Coefficients or Variation of Parameters, in our somewhat restricted circumstances: We have learned how to find <u>all</u> solutions of an equation like $\alpha y'' + \beta y" + \gamma = f(x)$, or in theory higher order equations, which have (1) constant coefficients on y and its derivatives and (2) a zero for "the other side". The text typically calls the representation of those solutions to the homogeneous equation y_h. We have learned that if we can find any one solution to an equation which resembles that one but has a non-zero function on the other side, something like $\alpha y'' + \beta y" + \gamma = f(x)$, we can combine that with the general solution to the homogeneous equation to get all solutions to the non-homogeneous equation. The book writes y_p for any particular solution to $\alpha y'' + \beta y" + \gamma = f(x)$, so this says we can write $y = y_h + y_p$ to represent all solutions. The problem we still have to solve is finding that particular solution y_p. Since <u>any</u> solution, any function that solves $\alpha y'' + \beta y" + \gamma = f(x)$, is OK, any way of finding y_p is fine including a lucky guess. The first example done in the book, on page 875, is essentially done this way: When the text says "observe that $y =$ constant would do, provided ...", that amounts to a lucky guess. But since luck is not always with you, we need a more systematic way to find y_p, and that is what both Variation of Parameters and Undetermined Coefficients attempt to provide.

Suppose we start with an equation of the form $\alpha y'' + \beta y" + \gamma = f(x)$. If it happens that $f(x) = 0$, the homogeneous case, we know how to find the general solution by solving the associated characteristic equation $\alpha r^2 + \beta r + \gamma = 0$ and using the resulting values of r in exponential and trigonometric functions. We are looking for a particular solution, y_p, to the given equation where $f(x) \neq 0$.

If we are very lucky and $f(x)$ is particularly simple, and that it does not interact in bad ways with the solutions to the homogeneous equation, we may be able to find a solution y_p as follows. We will assume that $f(x)$ being simple means it is some combination of terms like e^{nx}, $\cos(kx)$, $\sin(kx)$, and polynomials $ax^2 + bx + c$. (Note that if both cosine and sine terms are present, the same kx must appear in each.) We construct a candidate for y_p as follows:

For a term in $f(x)$ which is a multiple of	If	Then use a term like
$\sin(kx)$ or $\cos(kx)$	ki is not a root of the characteristic equation ki is a root of the characteristic equation	$A\cos(kx) + B\sin(kx)$ $Ax\cos(kx) + Bx\sin(kx)$
e^{nx}	n is not a root of the characteristic equation n is a single root of the characteristic equation n is a double root of the characteristic equation	Ce^{nx} $Cx\,e^{nx}$ $Cx^2\,e^{nx}$
A polynomial ax^2+bx+c of degree at most 2	0 is not a root of the characteristic equation 0 is a single root of the characteristic equation 0 is a double root of the characteristic equation	a polynomial $Dx^2 + Ex + F$ of the same degree as $ax^2 + bx + c$ a polynomial $Dx^3 + Ex^2 + Fx$ of degree one more a polynomial $Dx^4 + Ex^3 + Fx^2$ of degree two more

How we use this is best shown by examples: Here is Example 2 from page 876 in our text, done with Undetermined Coefficients rather than Variation of Parameters. The task is to find the solutions of $y'' + 2y' - 3y = 6$. The characteristic equation is $r^2 + 2r - 3 = 0$, with roots $r = -3$ and $r = 1$. The right side of the equation in this case is just $f(x) = 6$, a polynomial of degree zero. Since 0 is not a root of the characteristic equation, we try for y_p a generic polynomial of degree zero, i.e. a constant: Let $y_p = F$ where F represents a constant yet to be determined. Then y_p' and y_p'' are both 0. Plugging this into the equation, for y_p to be a solution, we must have $0 + 0 - 3F = 6$. Hence $F = -2$ (a) is the only possibility, but (b) is good enough. (Note that this is exactly the particular solution arrived at by Variation of Parameters in the book: They would not have to be the same, since any other function which was obtained from this one by adding on a solution to the homogeneous equation would be another usable particular solution.) The complete solution is obtained by adding on the general solution $C_1e^{-3x} + C_2e^x$ to the homogeneous equation.

Another example: Solve $y'' - y' = 5e^x - \sin(2x)$. The characteristic equation is $r^2 - r = 0$, with roots $r = 0$ and $r = 1$. The terms in $f(x)$ are $5e^x$ and $-\sin(2x)$. Start with $5e^x$, which is a constant times e^{nx} where $n = 1$. Since 1 is a single root of the characteristic equation, we put the term $Cx\,e^x$ into y_p. Now $-\sin(2x)$ is a multiple of $\sin(kx)$ where $k = 2$, and 2 is not a root of the characteristic equation, so we include $A\cos(2x)+B\sin(2x)$ in our candidate solution. Thus we arrive at $y_p = A\cos(2x)+B\sin(2x)+Cx\,e^x$. Then $y_p' = -2A\sin(2x)+2B\cos(2x)+Ce^x+Cx\,e^x$, and $y_p'' = -4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x$. Putting these into the equation we get $-4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x+2A\sin(2x)-2B\cos(2x)-Ce^x-Cx\,e^x = 5e^x-\sin(2x)$. We collect together the terms from both sides with $\cos(2x)$ and get $-4A-2B = 0$. From the $\sin(2x)$ terms we get $-4B+2A = -1$. The xe^x terms cancel out (a consequence of the fact that 1 was a root of the characteristic equation. . .) and the e^x terms give $C = 5$. Solving we get $A = -1/10$ and $B = 1/5$. Hence our particular solution is $y_p = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x + 5xe^x$. Combining this with the general solution the homogeneous equation we get is $-\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x + 5xe^x + C_1 + C_2e^x$.

Note that this method does have limitations. As given here, for example, it would not apply to Problem 2 on page 877 in the text, $y'' + y = \tan x$, since the limited table I included above does not suggest what to use for y_p when $f(x)$ includes the tangent function. It will, however work for all other problems on that page and for the "story problems" on page 882.

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