Notes and Extra Problems
using Taylor's Theorem

Our text does not have many examples or problems illustrating error estimation for truncated Taylor (Maclaurin) series. The situation where you might need to do this is typically something like this: For whatever reason you need to replace some function by a polynomial which approximates it, and you need to control the error introduced by using the polynomial (only a finite number of terms) rather than a full series representation of the function. There are several common forms of this problem, e.g. (a) If I use a given polynomial, how close will the results be to the values of the function? (b) Given that I need some prescribed precision in my approximation, what polynomial can I use that is guaranteed to provide that precision?

We can produce a polynomial which might be a useful approximation to $f(x)$ in several ways, but there is one specific way that we consider here. Given a function $f(x)$ that is "nice enough" (typically this means it can be differentiated as many times as we want at some point a) we can produce its Taylor series at a. (If $a = 0$ the series is what we call the Maclaurin series.) Unless the function $f(x)$ was itself a polynomial (so that its derivatives would be zero after some point) the series will have infinitely many terms. We could arbitrarily pick some number n and use just the terms up through degree n, what the book calls the nth degree Taylor polynomial for $f(x)$. (We could also call it the nth degree Maclaurin polynomial for $f(x)$ in the case $a = 0$.) But clearly that choice of n must not be made quite so arbitrarily. If we choose $n = 0$, for example, we do get a very simple polynomial to work with (just the constant $f(a)$, i.e. we are approximating what might be a very complex graph with a horizontal straight line) but for x's other than a the polynomial might be far different from $f(x)$. (I.e. the graph might rapidly get far from the straight line as we move away from $x = a$.) Hence we would not get much accuracy. (Note that any Taylor polynomial is perfectly accurate at $x = a$. When we say we don't get much accuracy we mean the accuracy is not sufficient at some other values of x that we care about.) So we need to choose n with care.

Here is a setting in which this may make more sense. Suppose that you are working on a team designing a new video game, which will involve graphics on the video screen that move as the player proceeds through the game. As the game moves along, the viewpoint of the screen picture changes. So the coordinate system has to be rotated. (Remember section 8-8 in the text?) That involves trig functions. But, the design of the game has to use the minimal amount of computing hardware so as to achieve a low price point! So you can't assume there is a lot of computing power the game program can call upon. Every computer instruction that will be in the game will be something your team wrote, not coming from some monstrous library of math functions. So when you need to use $\cos x$, for example, you have to know how to compute it using simple add/subtract/multiply/divide commands. Now in the video game the image on screen does not need to be perfectly accurate: The resolution of the monitor, limits in the player's visual acuity, etc., imply that dots making up the image on the screen need only be placed with a certain accuracy, not exactly. So you can get by with computing $\cos x$ to some accuracy depending on the needed screen precision. (A good thing: Although the processor in the game may be fast it is certainly not infinitely fast, so summing all the terms in a series is not an option!) Thus you conclude (a) I need to calculate $\cos x$, (b) I will need to do this for x's in some range $\alpha < x < \beta$ (which might be a whole circle like $0 < x < 2\pi$ but if you can restrict it you may be able to use a simpler calculation and save needed resources), and (c) you need to get the answers accurate to within some tolerance ϵ. If the game goes slowly enough around corners, so that each individual screen rotation is at most $30°$, you might find you needed to compute $\cos x$ for $-\frac{\Pi}{6} < x < \frac{\Pi}{6}$ and that you needed the result to be within 0.001 (to pick a number) of the actual value of $\cos x$. (I have know people who had to do almost exactly this in a slightly different setting, constructing special purpose computers for navigating in space.)

Taking the hypothetical problem above as an example, we know that the Maclaurin series for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$. The 0th degree polynomial approximation to $\cos x$ we get from this is just $\cos x \approx 1$, the 2th degree polynomial is $1 - \frac{x^2}{2!}$, etc. We need to decide what degree would insure

accuracy of 0.001 throughout the interval $(-\frac{\pi}{6}, \frac{\pi}{6})$. We might say "I'll bet that the 1000th degree polynomial would be good enough," but (even assuming that we were right) that would probably take a lot of extra computing. So we want to find some n such that (a) the nth degree polynomial can be shown to be good enough, but (b) n is not larger than it needs to be. We may not completely satisfy (b) but we will try to make n at least not *much* larger than the minimum it could be to achieve (a).

We have a couple of tools we may be able to use, one of which the book presents in two different forms. These are: (i) Taylor's Theorem as given in the text on page 792, where $R_n(x, a)$ (given there as an integral) tells how much our approximation might differ from the actual value of $\cos x$; (ii) The variation of this theorem where the remainder term $R_n(x, a)$ is given in the form on page 795, labelled as Lagrange's Form of the Remainder; (iii) the Alternating Series Estimation Theorem given on page 783. (Another tool would be the Estimation of Remainders by Integrals starting on page 769, but I will not consider that further since we skipped it in class.)

When the Alternating Series Estimation process applies, it is frequently the easiest to use, so I check that first. Since even powers of x cannot be negative, the series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ is an alternating series for any x. For any fixed x we can compare adjacent terms a_n and a_{n+1} in the series: If we compute their ratio

$$\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{2n!}} = \frac{x^2}{(2n+1)(2n+2)}$$

we see that the terms are decreasing. (Note that x is not changing in that expression so it shows that each term is the previous one multiplied by a number less than 1.) Hence we have that the difference between the sum of just the first n terms of the series, which will become our polynomial, and the whole series which would give the actual value $\cos x$, is in size at most as the first term we don't include. Since we need to make sure that difference is at most 0.001 we can use terms so that the first one left out is less than that number. But a term like $\frac{x^{2n+2}}{(2n+2)!}$ depends on x. So we have to make sure that this is less than 0.001 for any x in the interval $(-\frac{\pi}{6}, \frac{\pi}{6})$. The largest that x^{2n+2} can get on that interval is for x's near the ends of the interval: We can safely say that for any x in the interval, $x^{2n+2} < \left(\frac{\pi}{6}\right)^{2n+2}$. But what is $\left(\frac{\pi}{6}\right)^{2n+2}$? Here we can be somewhat creative: We want to find a good value of n but we can get by without knowing our value is the best possible. We can simplify our arithmetic if we note that $\pi < 6$ so $\frac{\pi}{6} < 1$, so $\left(\frac{\pi}{6}\right)^{2n+2} < 1^{2n+2} = 1$. Hence what we need to make sure of, that we choose n so that $\frac{x^{2n+2}}{(2n+2)!} < 0.001$ for all the relevant x's, can be guaranteed if we make $\frac{1}{(2n+2)!} < 0.001$. Now we try different factorials: $6! = 720$, and $7! = 5040$, so if we make $2n + 2$ at least 7 we will have $(2n + 2)! > 1000$ and so $\frac{1}{(2n+2)!} < 0.001$. There is no whole number n that makes $2n + 2 = 7$, that would require $n = 2\frac{1}{2}$, so we use the next higher value of n. So we can say that using $2n + 2 = 8$ ($n = 3$) we get the needed accuracy. This says we use as our polynomial the terms up to but not including the one with $\frac{x^8}{8!}$, since what we were looking at was the first term omitted. So the polynomial $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$ is what we use for our approximation.

In this case the series really was an alternating series with decreasing terms, so the alternating series estimation process worked. I will still go ahead and use Taylor's theorem so that you can compare the results and also the effort involved. For other functions you may have to use Taylor's theorem in one or the other form: Suppose the function to be approximated had been $\sin x$. In that case the terms of the Maclaurin series have odd powers of x, which changes the signs in the series when we use a negative x. For other series the terms might not even look like they have alternating signs, or might not be consistently decreasing in size.

I personally prefer almost always to use the Lagrange form of the remainder term, the one given as equation (15) on page 795 in our text. In our example, $a = 0$. So we have that the difference between the sum of the terms of the series through the one with x^n and the full sum is $R_n(x, 0) = f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!}$ where the function $f(x) = \cos x$ and c is some number between x and 0. We can't generally determine c. What we have to do is to choose n so that this expression gives at most 0.001 no matter what x is chosen in $(-\frac{\pi}{6}, \frac{\pi}{6})$ and no matter what value c takes between x and 0. Since $f(x) = \cos x$, all of its derivatives are either $\pm \cos x$ or $\pm \sin x$, so no matter what value c has we can say that $f^{(n+1)}(c)$ is

between -1 and 1. Hence

$$|R_n(x,0)| = \left| f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!} \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \right|.$$

But just as in the version using alternating series estimation, for the x's in our interval we can say $\left| x^{n+1} \right| < 1^{n+1} = 1$, so we have

$$|R_n(x,0)| \leq \left| \frac{x^{n+1}}{(n+1)!} \right| \leq \frac{1}{(n+1)!}.$$

So again we can choose n so as to make $\frac{1}{(n+1)!} < 0.001$, satisfied when we get to $\frac{1}{7!} < 0.001$, and again we use the terms through $\frac{x^6}{6!}$, so we use the same polynomial as before.

It can happen that one theorem turns out to guarantee sufficient precision using a smaller n than what you get from one of the other theorems. So if you really need to get the best possible results you should try all possibilities. But for this class all we will require of you is that you find an answer that can be justified using one of these theorems, not the best possible answer.

As mentioned in the first paragraph of this note, there are several variants of this form of problem: The others are usually easier, so I have illustrated the "worst case". One other form is to give you the polynomial, i.e. tell you we use the terms up through some specified degree, and an interval of x's, and ask you to tell how accurate an approximation the polynomial will give on that inteval, or ask you to show that it at least meets some specified accuracy. A different form is to give you a polynomial and an accuracy and ask what x's may be used in the polynomial and be sure of meeting the accuracy requirement. Each of these uses the same kind of reasoning shown above. Even easier problems ask you just to work with one value of x, e.g. the first problem below.

Additional problems to work on using this material:

(a) Determine how many terms of the Maclaurin series for e^x are needed to estimate $e^{0.1}$ within 0.00001.

(b) Find the range of x values for which $x - \frac{x^3}{6} + \frac{x^5}{120}$ gives a number within 0.0001 of $\sin x$.

(c) Find the 4th degree Taylor polynomial for $\cos x$ at $a = \frac{\pi}{3}$. Use some form of Taylor's theorem to estimate the accuracy when this polynomial is used to approximate $\cos x$, where x ranges over $0 \leq x \leq \frac{2\pi}{3}$.

(d) How many terms of the Maclaurin series for $\ln(1+x)$ are needed to estimate $\ln 1.4$ within 0.0001?

(e) Use a polynomial to approximate $\int_0^1 \sqrt{1+x^4}\,dx$ to two decimal places.

(f) For $f(x) = \sqrt{x}$ find the fourth degree Taylor polynomial at $a = 1$ approximating $f(x)$. Find an estimate of the error involved in using this approximation for $0.9 \leq x \leq 1.1$.

(g) Use a polynomial approximation to find $\cos 69°$ to five decimal places.