Answers to Extra Problems on Differential Equations

1. $e^{y-2} - \frac{dy}{dx}e^{x+2y} = 0$ with $y(0) = -2$

   You can separate the variables in this equation and get the general solution as $y(x) = \ln|C - e^{-x}| - 2$. When you use the initial condition to solve for C you get $C = 2$, so the solution is $y(x) = \ln|2 - e^{-x}| - 2$.

2. $(x+1)\frac{dy}{dx} + 2y = x$ with $y(0) = 1$

   Rearranging the equation as $\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x}{x+1}$ we see this as first order linear with $P(x) = \frac{2}{x+1}$ and $Q(x) = \frac{x}{x+1}$. Straightforward application of the standard technique for first order linear gives us $y(x) = \frac{1}{(x+1)^2}\left(\frac{x^3}{3} + \frac{x^2}{2} + C\right)$. Using $y(0) = 1$ gives $C = 1$.

3. $x\frac{dy}{dx} + 2y = x^2 + 1$ with $y(1) = 1$

   Again this is first order linear, with $P(x) = \frac{2}{x}$ and $Q(x) = \frac{x^2+1}{x}$. We get for the general solution $y = \frac{x^2}{4} + \frac{1}{2} + \frac{C}{x^2}$, and using $y(1) = 1$ gives $C = \frac{1}{4}$.

4. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ with $y(0) = 2$ and $y'(0) = -2$

   This is second order with constant coefficients, and homogeneous. The characteristic equation is $r^2 - 4r + 3 = 0$ which yields $r = 3$ and $r = 1$. Thus the general solution is $y = C_1e^{3x} + C_2e^x$. Using $y(0) = 2$ we get $C_1 + C_2 = 2$, and $y'(0) = -2$ gives $3C_1 + C_2 = -2$. Solving these gives $C_1 = -2$ and $C_2 = 4$, so the solution is $y(x) = -2e^{3x} + 4e^x$.

5. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ with $y(0) = 0$ and $y'(0) = 7$

   Again this is second order, constant coefficients, homogeneous. The characteristic equation $r^2 + 4r + 4 = 0$ factors as $(r+2)^2 = 0$ so the solution is $r = -2$, repeated. Thus the general solution to the differential equation is $y = C_1e^{-2x} + C_2xe^{-2x}$. Using $y(0) = 0$ we get immediately that $C_1 = 0$, so the solution simplifies to $y = C_2xe^{-2x}$. Then $y' = C_2e^{-2x} - 2C_2xe^{-2x}$, and $y'(0) = 7$ gives $C_2 = 7$. So the solution is $y(x) = 7xe^{-2x}$.

6. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x$ with $y(0) = 1$ and $y'(0) = -3$

   This is still second order, constant coefficients, but no longer homogeneous. We first solve the homogeneous equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$. The characteristic equation $r^2 + 2r = 0$ gives $r = 0$ and $r = -2$, so the general solution to this homogeneous equation is $y_h = C_1e^{0x} + C_2e^{-2x} = C_1 + C_2e^{-2x}$.

   Using the method of undetermined coefficients to find a particular solution y_p to the original equation, we see the right hand side $4x$ is a polynomial of degree one. Since zero is a (single) root of the characteristic equation we use $Ex^2 + Fx$ for y_p. (We can ignore the possibility of a cubic term Dx^3 since we only have to go to degree one higher than that of the polynomial $4x$.) Using $y_p = Ex^2 + Fx$ we have $y_p' = 2Ex + F$ and $y_p'' = 2E$. Putting these into the original differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x$ we get $2E + 2(2Ex + F) = 4x$ or $4Ex + (2E + 2F) = 4x$. From the terms with x we get $4E = 4$, $E = 1$, and then from the constant terms we get $2E + 2F = 0$ with $E = 1$ so $F = -1$. Thus $y_p = x^2 - x$.

   Now we combine y_h and y_p to get $y(x) = C_1 + C_2e^{-2x} + x^2 - x$ as the general solution to the differential equation, and we have to find the values of C_1 and C_2 to make this fit the initial conditions. Putting in $x = 0$ gives $y(0) = C_1 + C_2$ so this must be 1. Taking the derivative we have $y'(x) = -2C_2e^{-2x} + 2x - 1$ so $y'(0) = -2C_2 - 1$, which must give -3, so $C_2 = 1$. Then from $C_1 + C_2 = 1$ we get $C_1 = 0$. So the solution is $y(x) = e^{-2x} + x^2 - x$.

7. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$ with $y(0) = -2$ and $y'(0) = -2$

   This time when we solve the homogeneous equation the characteristic equation gives $r = 2$ and $r = -1$. So the homogeneous solution is $y_h = C_1e^{2x} + C_2e^{-x}$.

   Since the right hand side $3e^{2x}$ is a multiple of e^{nx} for $n = 2$, and 2 is a single root of the characteristic equation, we use $y_p = Cxe^{2x}$. From this we get $y_p' = Ce^{2x} + 2Cxe^{2x}$ and $y_p'' = 4Ce^{2x} + 4Cxe^{2x}$. Putting these into the differential equation we get $4Ce^{2x} + 4Cxe^{2x} - Ce^2x - 2Cxe^{2x} - 2Cxe^{2x} = 3e^{2x}$ which simplifies to $3Ce^{2x} = 3e^{2x}$, or $C = 1$. Hence our particular solution is $y_p = xe^{2x}$.

   Combining, we get $y = C_1e^{2x} + C_2e^{-x} + xe^{2x}$ as the general solution. Now we use the initial conditions. We have $y' = 2C_1e^{2x} - C_2e^{-x} + e^{2x} + 2xe^{2x}$ so $y'(0) = 2C_1 - C_2 + 1$, forcing $2C_1 - C_2 + 1 = -2$ or $2C_1 - C_2 = -3$. From $y(0) = -2$ we get $C_1 + C_2 + -2$. Solving these two equations we get $C_1 = -\frac{5}{3}$ and $C_2 = -\frac{1}{3}$. Hence the solution we are after is $y(x) = -\frac{5}{3}e^{2x} - \frac{1}{3}e^{-x} + xe^{2x}$.

8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 4e^{-x}$ with $y(0) = 1$ and $y'(0) = -\frac{1}{2}$

   The characteristic equation for the homogeneous equation is $r^2 - 2r + 5 = 0$, so $r = \frac{2\pm\sqrt{4-20}}{2} = 1 \pm 2i$. Thus the homogeneous general solution is $y_h = e^x(C_1 \cos 2x + C_2 \sin 2x)$.

   Since the right hand side of the original equation, $4e^{-x}$, is a multiple of e^{nx} where $n = -1$ is not a root of the characteristic equation, we can use $y_p = Ce^{-x}$. That makes $y_p' = -Ce^{-x}$ and $y_p'' = Ce^{-x}$, and putting those in the differential equation gives $Ce^{-x} + 2Ce^{-x} + 5Ce^{-x} = 4e^{-x}$, from which we get $8C = 4$ and so $C = \frac{1}{2}$. Hence the general solution to the real equation is $y(x) = e^x(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{2}e^{-x}$.

   Now we use the initial conditions. We have $y(0) = C_1 + \frac{1}{2}$, and $y(0) = 1$, so $C_1 = \frac{1}{2}$. We get $y'(x) = e^x(C_1 \cos 2x + C_2 \sin 2x) + e^x(-2C_1 \cos 2x + 2C_2 \cos 2x) - \frac{1}{2}e^{-x}$, so $y'(0) = C_1 + 2C_2 - \frac{1}{2}$, and since $y'(0) = -\frac{1}{2}$ we get the equation $C_1 + 2C_2 - \frac{1}{2} = -\frac{1}{2}$. Solving for C_2 (we already know C_1) we get $C_2 = -\frac{1}{4}$. Hence the solution is $y(x) = e^x(\frac{1}{2} \cos 2x - \frac{1}{4} \sin 2x) + \frac{1}{2}e^{-x}$.

Solve the following differential equations:

1. $e^x\frac{dy}{dx} + 2e^xy = 1$

   Multiply by e^{-x} to get $\frac{dy}{dx} + 2y = e^{-x}$. This is first order linear with $P(x) = 2$ and $Q(x) = e^{-x}$. Then the integrating factor $\rho = e^{\left(\int P(x)dx\right)}$ is e^{2x}, so the solution can be written as $y = e^{-2x}\int e^{2x}e^{-x}dx = e^{-2x}\int e^x dx = e^{-2x}(e^x + C)$. This can be slightly simplified as $y(x) = e^{-x} + Ce^{-2x}$.

2. $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

   Divide by x to put this in the form $\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$. This is first order linear with $P(x) = \frac{3}{x}$ and $Q(x) = \frac{\sin x}{x^3}$. We get $y = \frac{1}{x^3}\int x^3 \frac{\sin x}{x^3}\,dx = \frac{1}{x^3}\int \sin x\,dx = \frac{1}{x^3}(-\cos x + C) = -\frac{\cos x}{x^3} + \frac{C}{x^3}$.

3. $(x-1)^3\frac{dy}{dx} + 4(x-1)^2\,y = x + 1$

   After dividing by $(x-1)^3$ we get $\frac{dy}{dx} + \frac{4}{x-1}y = \frac{x+1}{(x-1)^3}$ which is first order linear in standard form. The integrating factor $\rho(x)$ works out to be $(x-1)^4$ so we get $y = \frac{1}{(x-1)^4}\int(x-1)(x+1)dx = \frac{1}{(x-1)^4}\int(x^2-1)dx = \frac{1}{(x-1)^4}\left(\frac{x^3}{3} - x + C\right)$.

4. $\frac{d^2y}{dx^2} + y = \cos x$

   The homogeneous equation $y'' + y = 0$ has characteristic roots $r = \pm i$ and solutions $C_1 \cos x + C_2 \sin x$. The right hand side of the real equation, $\cos x$, is of the form $\cos kx$ where $k = 1$. Since $1\,i$ is a root of the characteristic equation we use $y_p = Ax\cos x + Bx\sin x$. Then $y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x$ and $y_p'' = -Ax\cos x - Bx\sin x + 2B\cos x - 2A\sin x$. Putting these into the differential equation we get $-Ax\cos x - Bx\sin x + 2B\cos x - 2A\sin x + Ax\cos x + Bx\sin x = \cos x$. From this we can read off that $A = 0$ and $2B = 1$, so $B = \frac{1}{2}$. Thus the solution is $C_1 \cos x + C_2 \sin x + \frac{1}{2}x\sin x$.

5. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = \sin x$

   The characteristic equation $r^2 - r = 0$ for the homogeneous equation has roots $r = 0$ and $r = 1$. Using the fact that $e^{0x} = 1$ we get the general solution to the homogeneous equation as $y_h = C_1 + C_2e^x$.

   The right hand side of the non-homogeneous equation, $\sin x$, is of the form $\sin kx$ where $k = 1$. While 1 is a root of the characteristic equation, $i1$ is not, so we use $y_p = A\cos x + B\sin x$. Then $y_p' = -A\cos x + B\sin x$ and $y_p'' = -A\cos x - B\sin x$. In the differential equation these give us $-A\cos x - B\sin x + A\sin x - B\cos x = \sin x$, or $-(A+B)\cos x + (A-B)\sin x = \sin x$. Thus $A + B = 0$ and $A - B = 1$, from which we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. The solution is $y(x) = C_1 + C_2e^x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$.

6. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 20\cos x$

   Solving the homogeneous version of the equation we find characteristic roots $r = 2$ and $r = -1$, so $y_h = C_1e^{2x} + C_2e^{-1}$.

   Since the right hand side $20\cos x$ is a multiple of $\cos kx$ where $k = 1$ and $1i$ is not a root of the characteristic equation, we use $y_p = A\cos x + B\sin x$. Taking derivatives and plugging into the original equation we get $-A\cos x - B\sin x + A\sin x - B\sin x - 2A\cos x - 2B\sin x = 20\cos x$ which simplifies to $(-3A - B)\cos x + (A - 3B)\sin x = 20\cos x$. Solving $-3A - B = 20$ and $A - 3B = 0$ we get $A = -6$ and $B = -2$. Putting the pieces together gives $C_1e^{2x} + C_2e^{-x} - 6\cos x - 2\sin x$ as the solution.