Trends in Mathematics, Geometry, from the ancient Greeks to POINCARÉ and PERELMAN

Jean-Pierre BOURGUIGNON
(CNRS-IHÉS)

Subtitle

The Ever Repeating Story of inspired mathematical insights on space by Astronomers, Physicists, ...

Pretext to show

- ✔ *Mathematics as a machine that creates concepts and solves problems,*
- ✔ *how Geometry evolved from uniqueness to abundance,*
- ✔ *how the philosophy behind it evolved,*
- ✔ *still exhibiting the remarkable unity of Mathematics,*
- ✔ *documents of historical importance,*
- ✔ *and a nice gallery of portraits.*

I. From Antiquity on

There have been contributions to the understanding of space by several civilisations (Egypt, Mesopotamia, China).

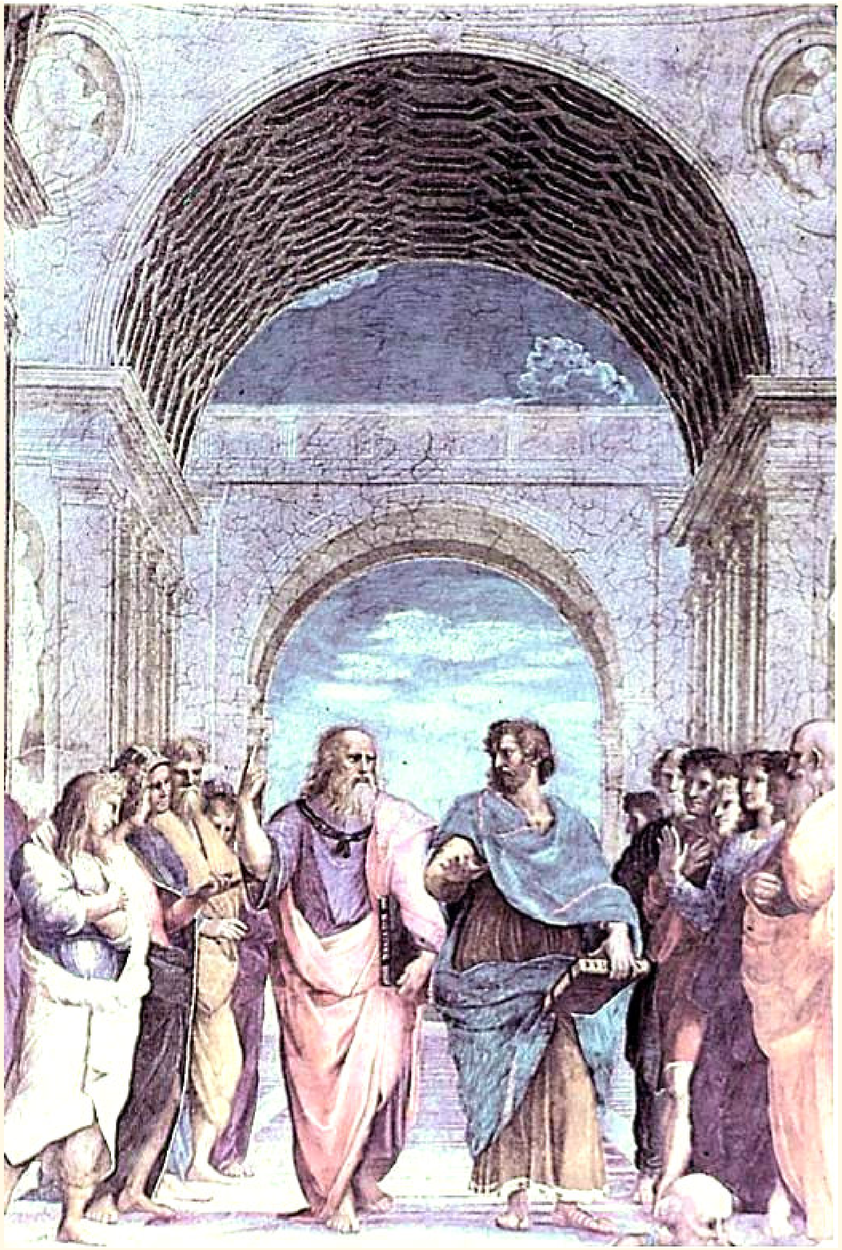
We will here concentrate on the Greek one :

- ✔ *for its long lasting contribution,*
- ✔ *for the variety of visions expressed,*
- ✔ *for the model of intellectual creation it provided.*

JPB-University Lecture

Madison, April 9, 2008

Aristotle
and Plato



The Academy

« *Non geometers are not allowed here* »

Euclid's Elements:

- ✔ *probably the non religious book with the longest influence in the History of Mankind,*
- ✔ *provides a model of space and a method to analyse it and to understand it,*
- ✔ *establishes the axiomatic method,*
- ✔ *is also a source of problems as the one posed by the Vth postulate, also called the parallel postulate.*

« *If two straight lines intersect a third one, while forming internal angles whose sum is less than a flat angle, then they intersect on the side of these angles provided one extends them far enough.* »

- ✔ equivalent formulation : « *from a point not on a line passes one and only one parallel to this line* » ;
- ✔ another equivalent formulation : « *in a triangle, angles sum up to a flat angle* »
- ✔ the only postulate involving infinity.

II. How Euclidean Geometry Survived Several Scientific Revolutions

Several major contributions changed our understanding of space:
That of Galileo GALILEI

Galileo GALILEI

Several major contributions changed our understanding of Space:
That of Galileo GALILEI, who

- ✔ *in the essay Il Saggiatore, states that « the great book of the Universe is written in the langage of Geometry » ;*

- ✔ *also states that the state of rest and the uniform linear movement are equivalent from a mechanical point of view ;*

That of Johannes KEPLER

Johannes KEPLER

That of Johannes KEPLER who

✔ *propose precise laws for the motions of planets:*

✿ *planets move along ellipses around the Sun for which it is one of the foci;*

✿ *the segment between the Sun and the planet sweeps equal areas in equal times;*

✿ *there is an algebraic relation between the length of the large axis of the ellipse and the period.*

That of René DESCARTES

René DESCARTES

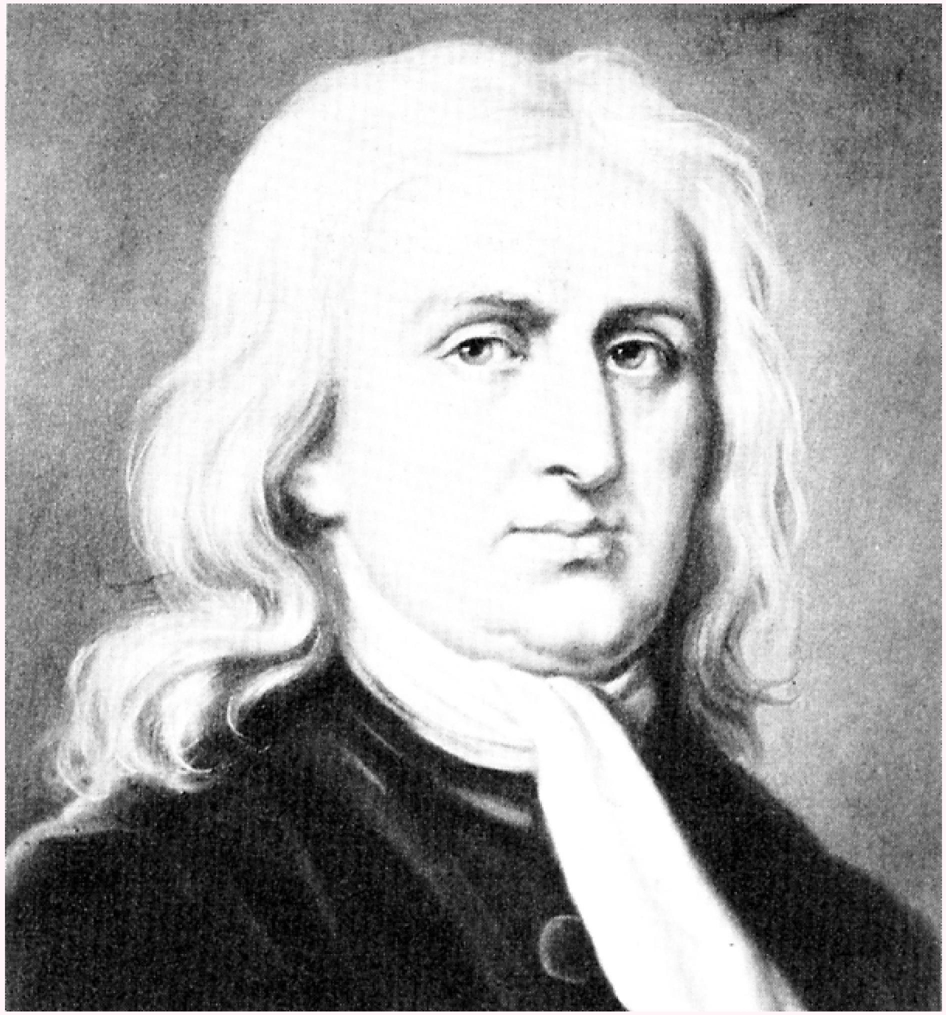
That of René DESCARTES who

✔ *through his Analytic Geometry, mixed numbers and geometric figures, a true revolution :*

✿ *that broadens considerably figures that can be considered;*

✿ *lays the foundations for a systematic analytic handling of geometric problems;*

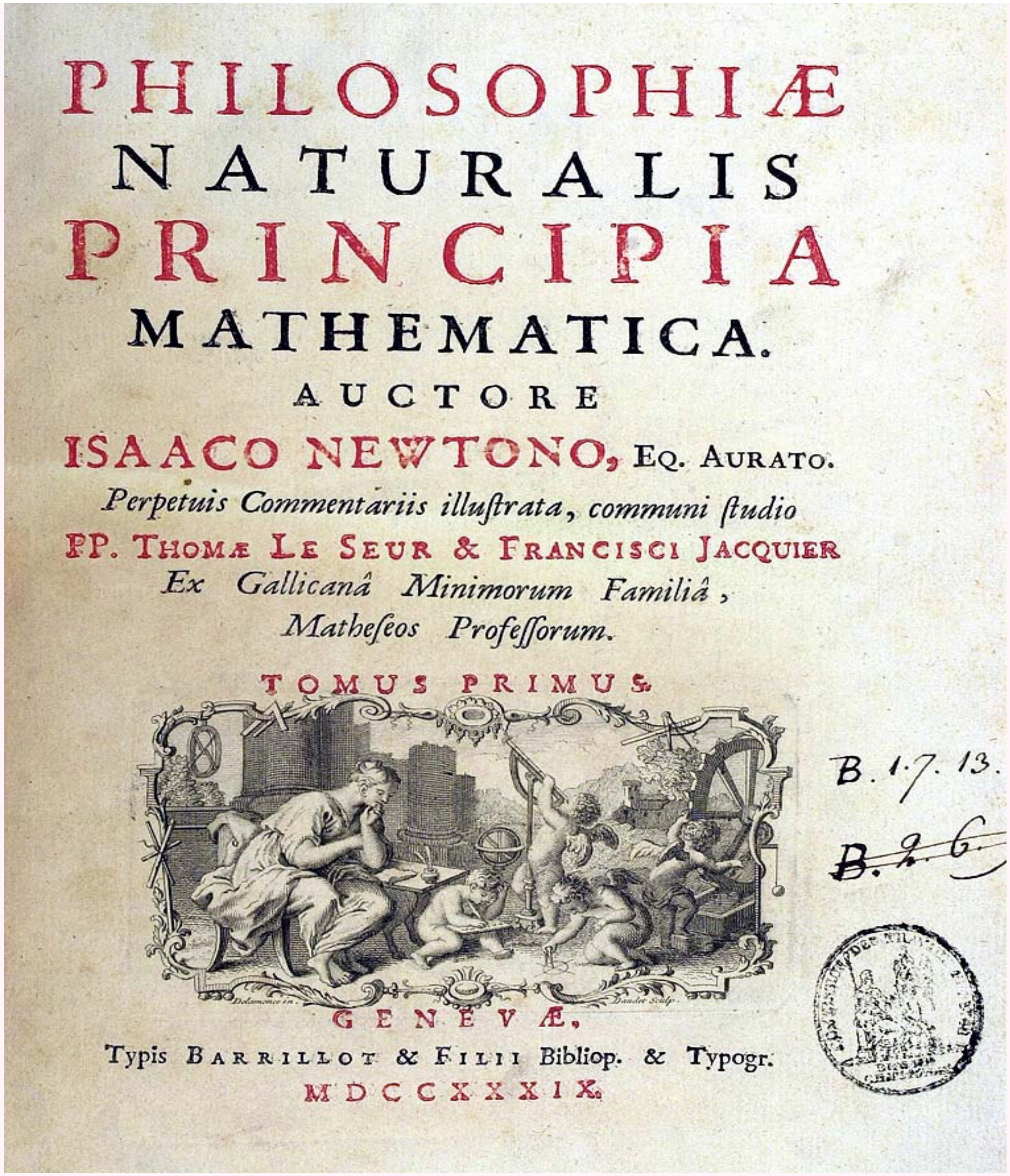
That of Isaac NEWTON



Isaac NEWTON

That of Isaac NEWTON who

✔ *wrote another most influential book, the Philosophia Naturalis Principia Mathematica in 1687*



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
AUCTORE
ISAACO NEWTONO, EQ. AURATO.
Perpetuis Commentariis illustrata, communi studio
PP. THOMÆ LE SEUR & FRANCISCI JACQUIER
Ex Gallicanâ Minimorum Familiâ,
Matheseos Professorum.

TOMUS PRIMUS.

GENEVÆ,
Typis BARRILLOT & FILII Bibliop. & Typogr.
MDCCXXXIX.

That of Isaac NEWTON who

- ✔ *wrote another most influential book, the Philosophia Naturalis Principia Mathematica in 1687 that mimicks Euclid's Elements*

PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA.

DEFINITIONES.

DEFINITIO I. [a]

Quantitas Materiæ eſt menſura ejuſdem orta ex illius Denſitate & Magnitudine conjunctim.

AER, denſitate duplicata, in ſpatio etiam duplicato fit quadruplus; in triplicato ſextuplus. Idem intellige de Nive & Pulveribus per compreſſionem vel liquefac-

Tom. I. A tionem

Licet primæ definitiones NEWTONIANÆ *vix aliquam poſtulare videantur explicationem; in ipſo tamen operis noſtri limine, nonnulla levioris momenti præmittenda judicamus, quæ ad majora viam ſternunt. Prima quæ in poſterùm ſæpiùs recurrent Mechanices principia interſerere non abs re erit, tùm ut Lectorum labori parcamus, tùm ut magis continua ſervetur noſtrarum demonſtrationum ſeries.*

(a) 1. Materia eſt ſubſtantia trinâ dimenſione prædita, ſolida ſeu impenetrabilis, mobilis, diviſibilis. Spatium purum eſt illa immenſa, penetrabilis, ſui ubique

That of Isaac NEWTON who

✔ *wrote another most influential book, the Philosophia Naturalis Principia Mathematica in 1687 that mimicks Euclid's Elements,*

✔ *and in which he does three things in the same course of development:*

✿ *he creates the Differential Calculus;*

✿ *he states the fundamental law of Mechanics;*

✿ *he states the law of Gravitation.*

NEWTON obtains KEPLER's laws in a deductive way

✔ from the fundamental law of Mechanics and

✔ from the form he proposes for the gravitational interaction.

This confidence in Euclidean geometry was consolidated by Immanuel KANT who

- ✔ *in his « Critic of Pure Reason », another most influential book published in 1781*

Immanuel KANT

This confidence in Euclidean geometry was consolidated by Immanuel KANT

- ✔ *in his « Critic of Pure Reason », published in 1781 takes Euclidean Geometry and Newtonian Physics as a priori truths,*
- ✔ *and does not discuss Space and Time.*

III. The Emergence of New Geometries

At the same time, several mathematicians were working intensely again at proving Euclid's Vth postulate: one of them was Johann Heinrich LAMBERT

Johann Heinrich LAMBERT

At the same time, several mathematicians were working intensely again at proving Euclid's Vth postulate: one of them was Johan Heinrich LAMBERT. He came to the conclusion that geometries

- ✔ *violating the parallel postulate,*
- ✔ *and in which angles of a triangle would not sum up to a flat angle*

can exist.

As many of you know, there are three main figures to which we owe non-Euclidean geometries:

- ✔ the most famous is Carl-Friedrich GAUSS

Carl Friedrich GAUSS

DISQUISITIONES GENERALES

CIRCA

SUPERFICIES CURVAS

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE OBLATAE D. 8. OCTOB. 1827

COMMENTATIONES SOCIETATIS REGIAE SCIENTIARUM
GOTTINGENSIS RECENTIORES. VOL. VI. GOTTINGAE MDCCCXXVIII

GOTTINGAE
TYPIS DIETERICHIANIS
MDCCCXXVIII

As many of you know, there are three main figures to which we owe non-Euclidean geometries:

- ✔ the most famous is Carl-Friedrich GAUSS,
- ✔ but Nicolas LOBACHEWSKI is really the first one who formally made the critical step

Nicolas LOBACHEWSKI

Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Herausgegeben

von

A. L. Crelle.

Mit thätiger Beförderung hoher Königlich-Preufsischer Behörden.

Siebenzehnter Band.

In vier Heften.

Mit drei Kupfertafeln.

Berlin, 1837.
Bei G. Reimer.

Et se trouve à Paris chez Mr. Bachelier (successeur de Mme Ve Courcier), Libraire pour les Mathématiques etc. Quai des Augustins No. 55.

18.

Géométrie imaginaire.

(Par Mr. *N. Lobatschewsky*, recteur de l'université de Cazan.)

Il y a à peu près cinq ans que j'ai fait insérer dans un journal scientifique qui paraissait à Cazan, quelques articles sur les élémens de la géométrie. Après y avoir développé une nouvelle théorie des parallèles, j'ai tâché de prouver que rien n'autorise, si ce ne sont les observations directes, de supposer dans un triangle rectiligne la somme des angles égale à deux angles droits, et que la géométrie n'en peut pas moins exister, si non dans la nature, au moins dans l'analyse, lorsqu'on admet l'hypothèse de la somme des angles moindre que la démicirconférence du cercle. Dans les articles cités j'étais même parvenu, par des considérations toujours géométriques et ne m'appuyant que sur cette nouvelle hypothèse, à donner des équations fondamentales pour le rapport entre les côtés et les angles d'un triangle rectiligne; enfin j'ai donné aussi les expressions générales pour les élémens différentiels des lignes courbes, des surfaces et du volume des corps dans cette géométrie nouvelle que je veux nommer *imaginaire*. Cependant resserré alors dans les limites d'un journal, je ne crois pas avoir traité ce sujet avec tout le détail nécessaire. Je m'apercois à présent que beaucoup de propositions que j'y ai annoncées sans en donner en même tems les démonstrations, et le peu de développement qu'on doit rémarquer d'abord dans des calculs fort longs et embarassants, n'ont peut être que trop contribué à rendre inintelligible tout mon travail et à jeter même du doute sur la vérité de ce que je voulais y énoncer. Mais si d'un côté je ne désirais revenir sur cette matière qu'en écrivant déjà d'après un plan

Geometrische Untersuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatschewsky,
Kaiserl. russ. wirkl. Staatsrathe und ord. Prof. der Mathematik
bei der Universität Kasan.

2. unveränderte Auflage.

Berlin

As many of you know, there are three main figures to which we owe non-Euclidean Geometries:

- ✔ the most famous is Carl-Friedrich GAUSS,
- ✔ but Nicolas LOBATCHEWSKI is really the first one who formally made the critical step,
- ✔ and Janosz BOLYAI should not be forgotten.

Janosz BOLYAI

APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

a veritate aut falsitate Axiomatis XI *Euclidei* (*a priori haud unquam decidenda*) *independentem:* adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

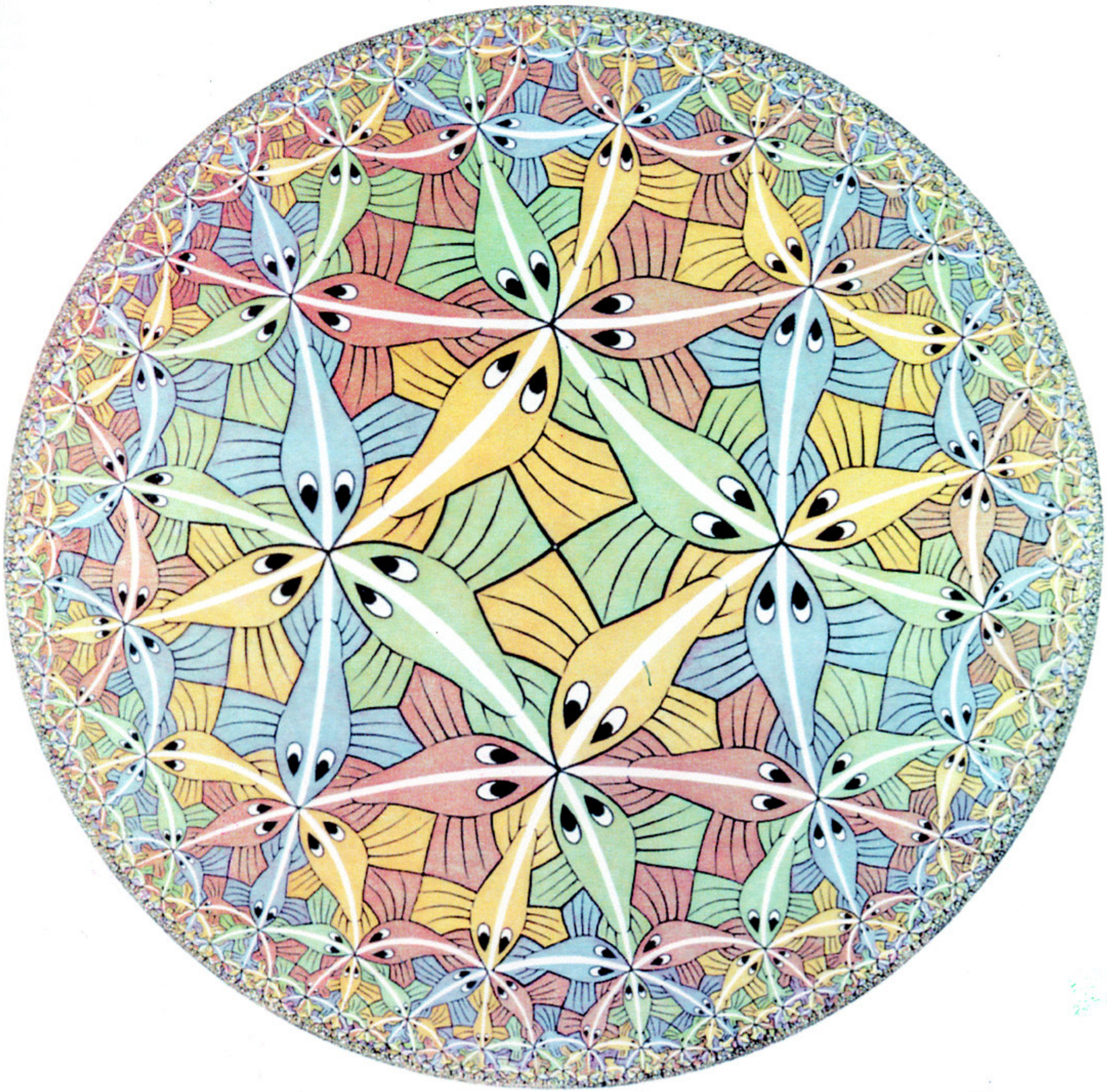
This new Geometry is called hyperbolic: as expected,

- ✔ *it violates the Vth postulate,*
- ✔ *in it triangles angles sum up to less than a flat angle.*

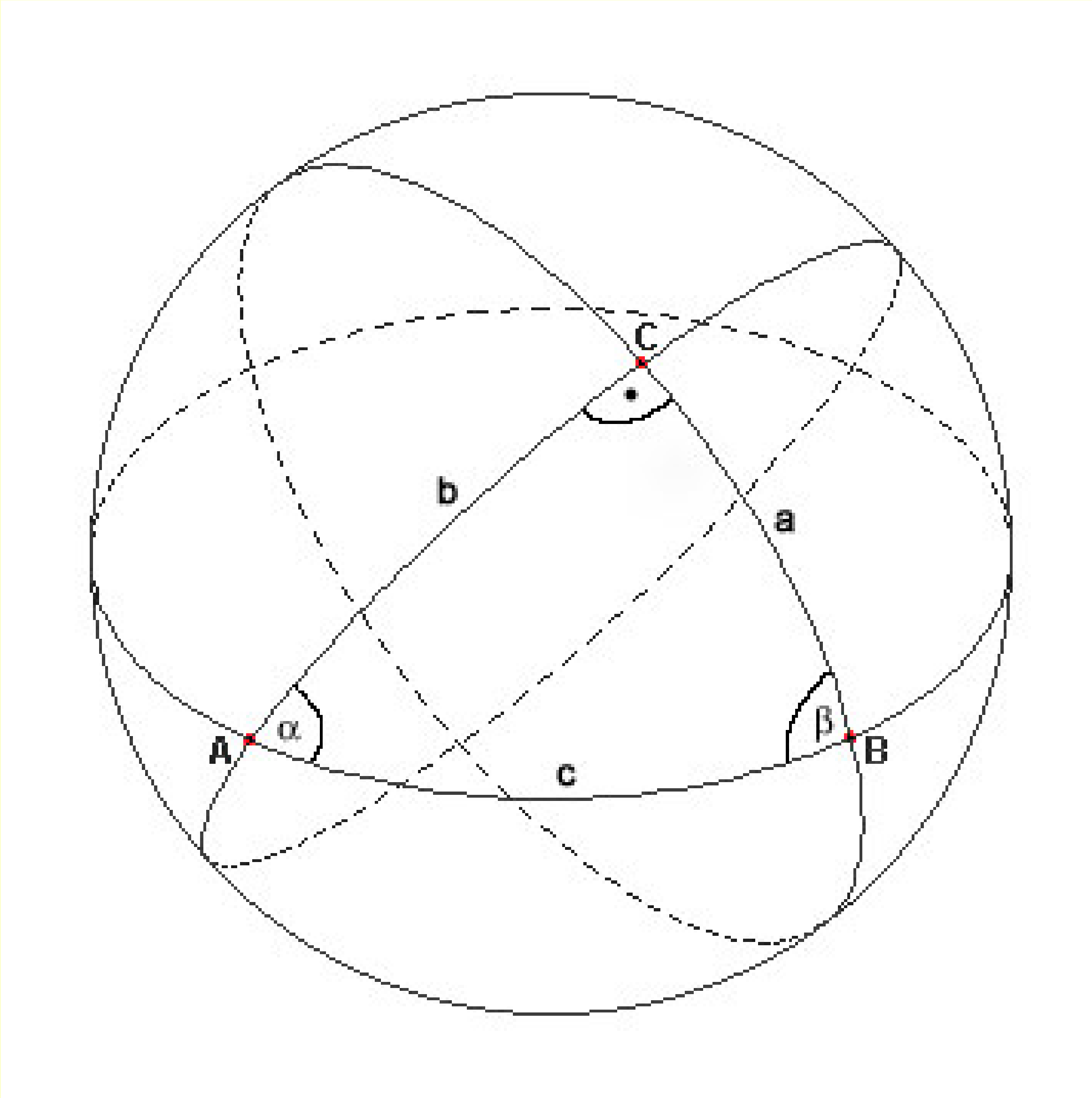
It took a long time for this new Geometry to be accepted.

Actually it is only when it had a Euclidean model that it really took ground.

Enrico BELTRAMI



The acceptance of hyperbolic Geometry led people to realize that they actually already knew a non-Euclidean Geometry, namely spherical Geometry, used and studied by astronomers for a long time:


C
b
a
α
A
β
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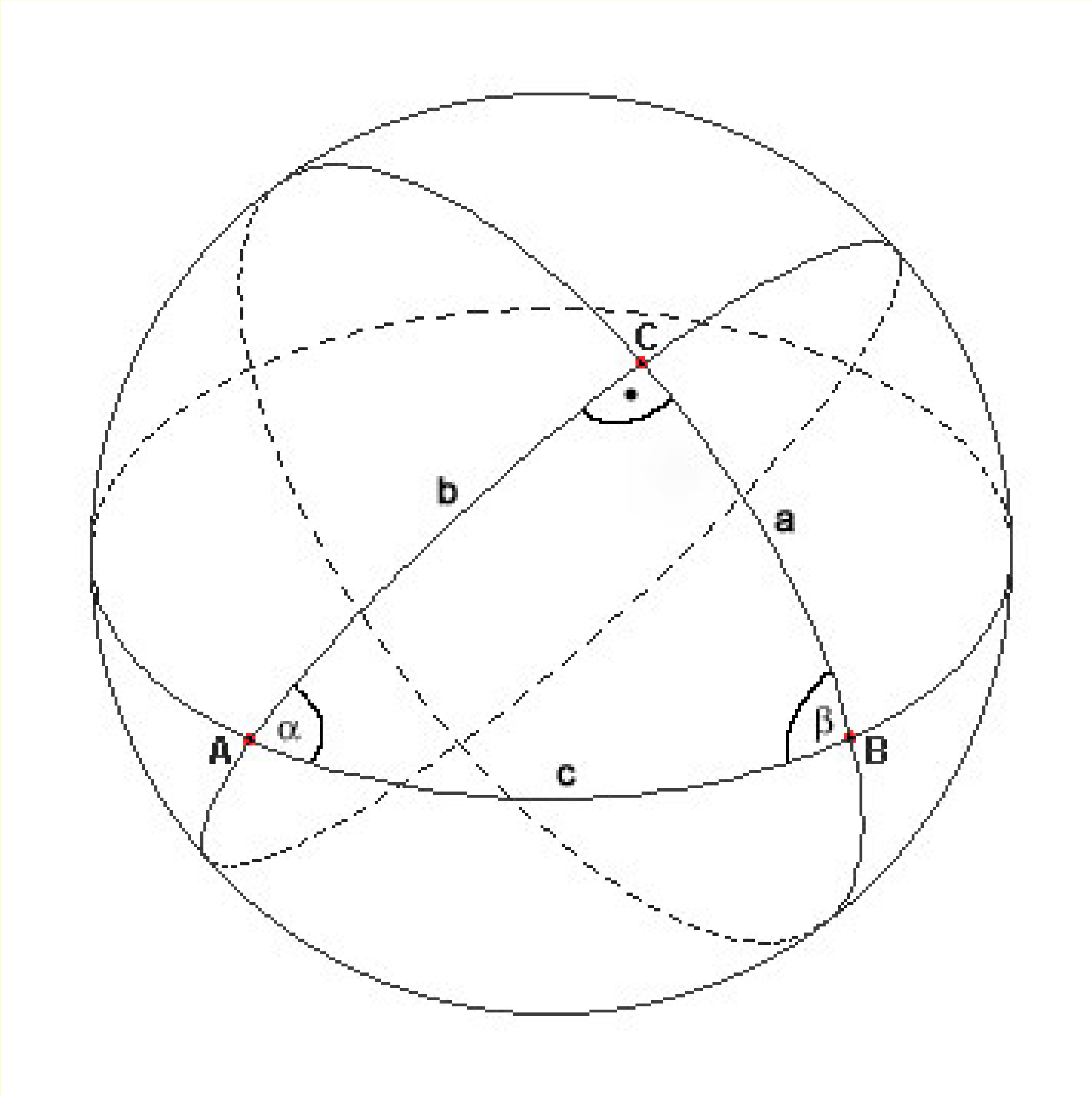
c

The acceptance of hyperbolic Geometry led people to the conclusion that they actually already knew a non-Euclidean Geometry, namely spherical Geometry, used and studied by astronomers for a long time:

- ✔ *it does violate the Vth postulate,*
- ✔ *in a spherical triangle, the sum of angles always exceeds a flat angle,*

and the excess is given by the area of the triangle.


C
b
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c

Before going any further, it is useful to think back to trigonometry:

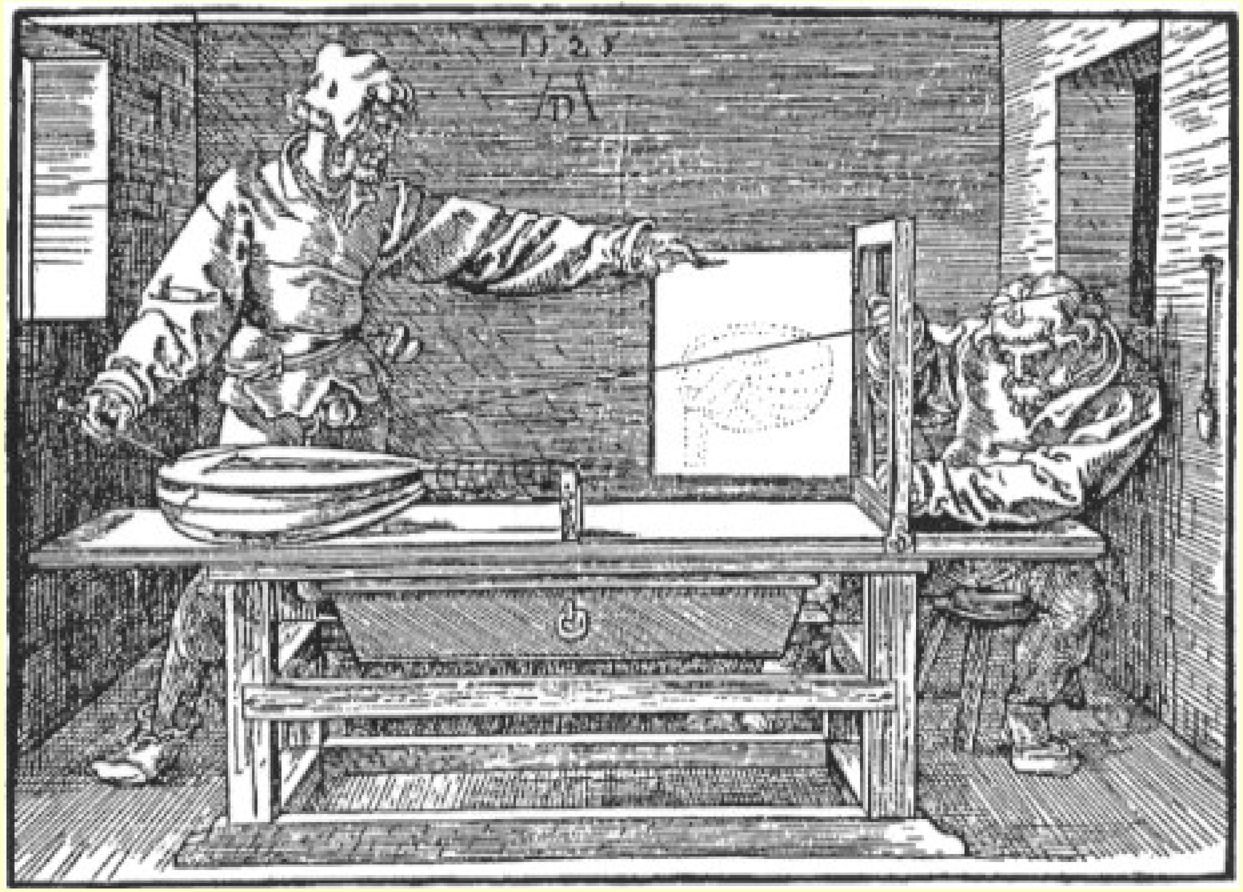
✔ actually its knowledge can concentrate all what one knows about Euclidean Geometry, to start with triangle formulas

$$a^2 = b^2 + c^2 - 2\, bc \cos \alpha \,;$$

✔ astronomers had been using spherical trigonometry for a long time, notably the formula for spherical triangles

$$\cos a = \cos b + \cos c - \sin b \sin c \cos \alpha \,.$$

Actually, there is another (very achieved form of Geometry) which had been around, namely projective Geometry:



Perspective according to A. DÜRRER

Actually, there is another (very achieved form of Geometry) which had been around, namely projective Geometry:

- ✔ several artists had discovered the rules to handle perspective, actually a piece of mathematics to handle the plane projection of 3-dimensional space;

- ✔ a key contribution for this came from Girard DESARGUES

Girard DESARGUES

Actually, there is another (very achieved form of Geometry) which had been around, namely projective Geometry:

- ✔ several artists have discovered the rules to handle perspective, actually a piece of mathematics to handle the plane projection of 3-dimensional space;
- ✔ a key contribution for this came from Girard DESARGUES ;
- ✔ but the modern view came in the XIXth century from Jean Victor PONCELET.

Jean Victor PONCELET

V. Other Major Steps in the Broadening of the Concept of Geometry

The next generalisation of Geometry came from the genius of Bernhard RIEMANN

Bernhard RIEMANN

The next generalisation of Geometry came from the genius of Bernhard RIEMANN :

- ✔ it came as a formal exercise on the occasion of his thesis defense;
- ✔ it is by now well established that it was inspired by the consideration he was giving at the time to the physical theory of Ether;
- ✔ « *Über die Hypothesen, welche der Geometrie zu Grunde liegen* » was published after his death in 1868.

Ueber die Hypothesen, welche der Geometrie zu Grunde liegen.

(Aus dem dreizehnten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen.)*)

Plan der Untersuchung.

Bekanntlich setzt die Geometrie sowohl den Begriff des Raumes, als die ersten Grundbegriffe für die Constructionen im Raume als etwas Gegebenes voraus. Sie giebt von ihnen nur Nominaldefinitionen, während die wesentlichen Bestimmungen in Form von Axiomen auftreten. Das Verhältniss dieser Voraussetzungen bleibt dabei im Dunkeln; man sieht weder ein, ob und in wie weit ihre Verbindung nothwendig, noch a priori, ob sie möglich ist.

Diese Dunkelheit wurde auch von **Euklid** bis auf **Legendre**, um den berühmtesten neueren Bearbeiter der Geometrie zu nennen, weder von den Mathematikern, noch von den Philosophen, welche sich damit beschäftigten, gehoben. Es hatte dies seinen Grund wohl darin, dass der allgemeine Begriff mehrfach ausgedehnter Grössen, unter welchem die Raumgrössen enthalten sind, ganz unbearbeitet blieb. Ich habe mir daher zunächst die Aufgabe gestellt, den Begriff einer mehrfach ausgedehnten Grösse aus allgemeinen Grössenbegriffen zu construiren. Es wird daraus hervorgehen, dass eine mehrfach ausgedehnte Grösse verschiedener Massverhältnisse fähig ist und der Raum also nur einen besonderen Fall einer dreifach ausgedehnten Grösse bildet. Hiervon aber ist eine nothwendige Folge, dass die Sätze der

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*) Diese Abhandlung ist am 10. Juni 1854 von dem Verfasser bei dem zum Zweck seiner Habilitation veranstalteten Colloquium mit der philosophischen Facultät zu Göttingen vorgelesen worden. Hieraus erklärt sich die Form der Darstellung, in welcher die analytischen Untersuchungen nur angedeutet werden konnten; einige Ausführungen derselben findet man in der Beantwortung der Pariser Preisaufgabe nebst den Anmerkungen zu derselben.

The new Geometry considered by RIEMANN encompassed greatly the previous ones:

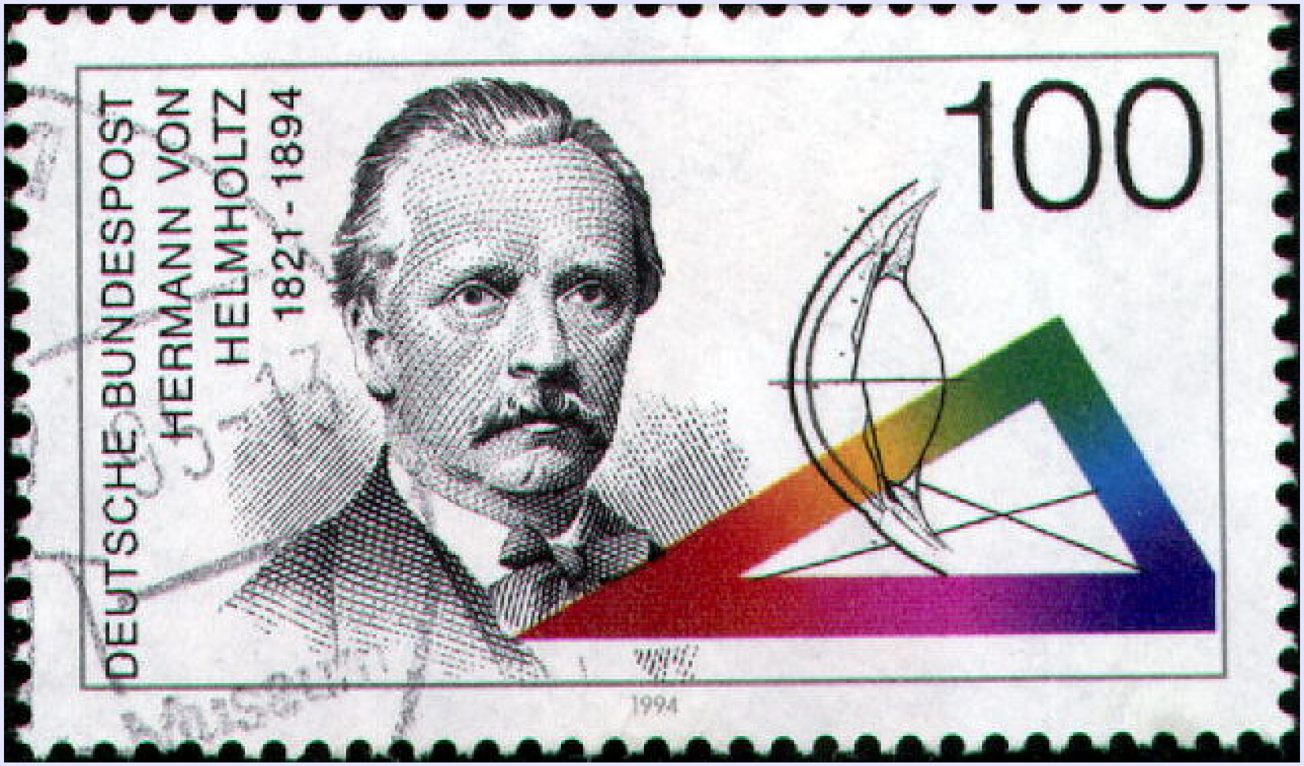
- ✔ it is founded on the variability of the line element,
- ✔ shortest paths are curves that generalise lines, whose collective behaviour is governed by the most important new feature introduced by RIEMANN, namely the curvature tensor

$$R_{ijk}^{\ \ \ l} \ .$$

At about the same time, another great mind

Hermann von HELMHOLTZ

Hermann von HELMHOLTZ


DEUTSCHE BUNDESPOST
HERMANN VON
HELMHOLTZ
1821 - 1894
100
1994

At about the same time, another great mind Hermann von HELMHOLTZ published:

- ✔ «*Über die Tatsachen, welche der Geometrie zu Grunde liegen* », Dieses Interesse musste um so allgemeiner und dauernder werden, als das folgende Jahr den Beweis brachte, wie notwendig die von Riemann angebahnte Verallgemeinerung mit dem Stande der mathematischen Wissenschaften verknüpft war. Helmholtz nämlich veröffentlichte in den „Heidelberger Jahrbüchern der Literatur“, sowie in den „Göttinger Nachrichten“ zwei Aufsätze [1], welche zeigten, dass er von einem ganz verschiedenartigen Gebiete aus durch analoge Betrachtungen zu ganz denselben mathematischen Ergebnissen und philosophischen Consequenzen gelangt war. Seinen Ausgangspunkt bildeten die Untersuchungen der physiologischen Optik „über die räumlichen Anschauungen im Gesichtsfelde, speziell das System der Farben und die Ausmessung des Gesichtsfeldes durch das Augenmaſs.“ Die Vergleichung dieser Mannigfaltigkeiten, die sich, wie wir genauer sehen werden, analytisch durch unserem Raumsystem analoge Bedingungen darstellen lassen, mit den Maſsbeziehungen unserer Geometrie, hatte ihn zu einer Fragestellung geführt, die der Formulirung Riemanns völlig entspricht. Dennoch zeigen die Arbeiten beider Forscher wesentliche formelle und sachliche Differenzen. Die erste besteht darin, dass Helmholtz in den genannten beiden ersten Abhandlungen lediglich die mathematische Frage behandelt: „Wieviel von den Sätzen der Geometrie hat objectiv giltigen Sinn; wieviel ist im Gegenteil nur Definition oder Folge aus Definitionen, oder von der Form der Darstellung abhängig?“ Die weitere Frage, woher unsere Kenntnis der Sätze von tatsächlicher Bedeutung stammt, die nicht mehr wie die erstere in das Bereich der exacten Wissenschaften gehört und von der jene Untersuchung ganz unabhängig ist, wird nur in so weit berührt, als die Ergebnisse auf sie hindeuten, dadurch allerdings ganz im Sinne des schon von Riemann behaupteten Empirismus beantwortet. Erst die neuere Abhandlung „über den Ursprung und die Bedeutung der geometrischen Axiome“ [2] zieht die philosophische Seite der Probleme ausdrücklich herbei. Bedeutsamer noch als diese charakteristische Enthaltsamkeit ist der sachliche Unterschied.

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[1] *„Ueber die tatsächlichen Grundlagen der Geometrie.“* Heidelberger Jahrbücher 1868, Nr. 46 u. 47 und *„Ueber die Tatsachen, die der Geometrie zum Grunde liegen.“* Göttinger Nachrichten 1868, Nr. 9.

[2] Helmholtz. *Populäre Vorträge.* Heft III. Braunschweig 1876. S. 21 f.

At about the same time, another great mind Hermann von HELMHOLTZ published:

- ✔ *«Über die Tatsachen, welche der Geometrie zu Grunde liegen », in which he stresses the importance of spaces for which isometries act transitively;*

- ✔ *it has been greatly ignored in the XXth century for reasons that would be interesting to study;*

- ✔ *more attention is usually given to some work of Felix KLEIN.*

Felix KLEIN

Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,

o. ö. Professor der Mathematik an der Universität Erlangen.

Programm

zum Eintritt in die philosophische Facultät und den Senat
der k. Friedrich-Alexanders-Universität
zu Erlangen.

Erlangen.

Verlag von Andreas Deichert.

1872.

In his « Erlangen Programm », KLEIN formulates what HELMHOLTZ had formulated before in a very sharp way:

- ✔ *a Geometry is the set of properties invariant by a group of transformations; he singles out spaces with enoug special vector fields, the infinitesimal isometries;*

- ✔ *a final end for uniqueness of Geometry;*

- ✔ *this brings Geometry close to Algebra, in particular to Group Theory, notably the Theory of Representations of Groups.*

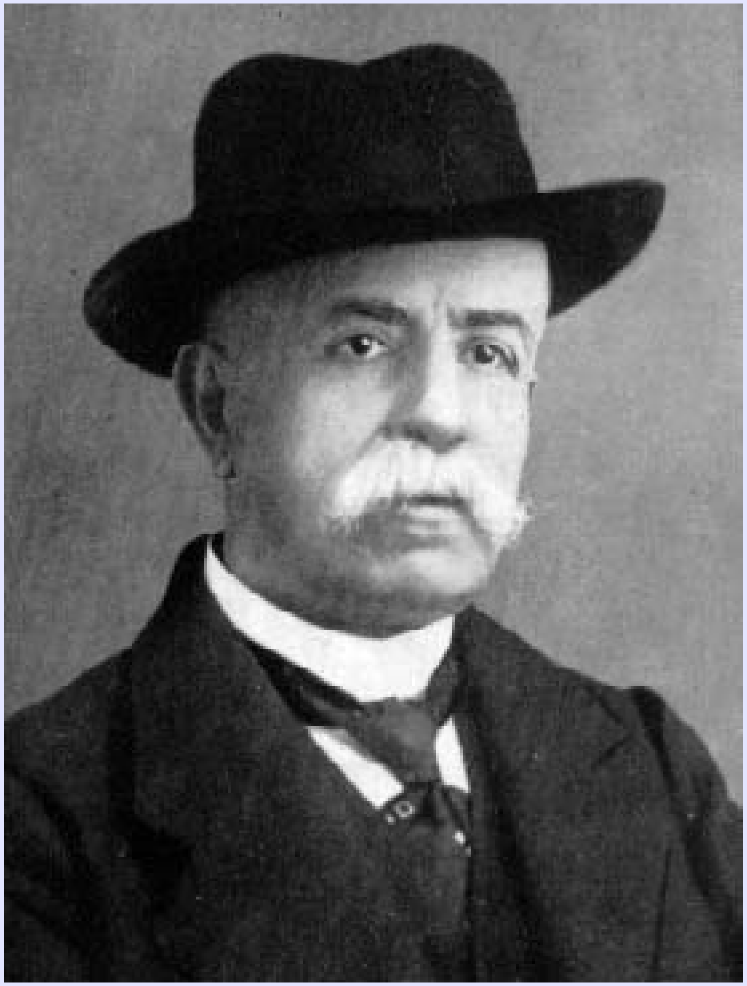
Others attempts went along at the same time, some incredibly visionary such as some work of William Kingdon CLIFFORD

William Kingdon CLIFFORD

Others attempts went along at the same time, some incredibly visionary such as some work of William Kingdon CLIFFORD

- ✔ *in « The Space Theory of Matter », published in 1870, he anticipated the rôle of curvature in the gravitational attraction.*

The key technical consolidation of notions of Riemannian Geometry was done in 1888 by Grigorio RICCI-CURBASTRO



Grigorio RICCI-CURBASTRO

He in particular understood fully the meaning of the covariant derivative and how it was uniquely determined by the metric, but gave to it another name!

In the later part of the XIXth century, new developments both in Algebra and in Geometry, came in particular in the study of continuous groups through the major work of Sophus LIE

Sophus LIE

The need for an axiomatic presentation of Euclidean Geometry in order to be on the safe side led David HILBERT to publish his « *Grundlagen der Geometrie* » in 1899.

David HILBERT

There was still an important step to be made widely clear, namely the rôle of these new Geometries in their relation with the world around, namely a philosophical question

- ✔ this was in particular done, with a wide impact on the public by Henri POINCARÉ,

Hennri POINCARÉ

Here is a quote from « *La Science et l'Hypothèse* » (1902):

- ✔ « *Geometric axioms are neither synthetic a priori statements or experimental facts. They are conventions*… »

  « *What should one think of the questions: Is Euclidean Geometry true? It does not make sense*.

  A Geomety cannot me more true than another: it can only be more efficient » .

Robert
DELAUNAY

« Eiffel Tower »

VI. Here Comes Physics again

At the same time, Physics was making great progress thanks in particular to the new understanding of Electromagnetism since the unification due to James Clark MAXWELL

James Clark MAXWELL

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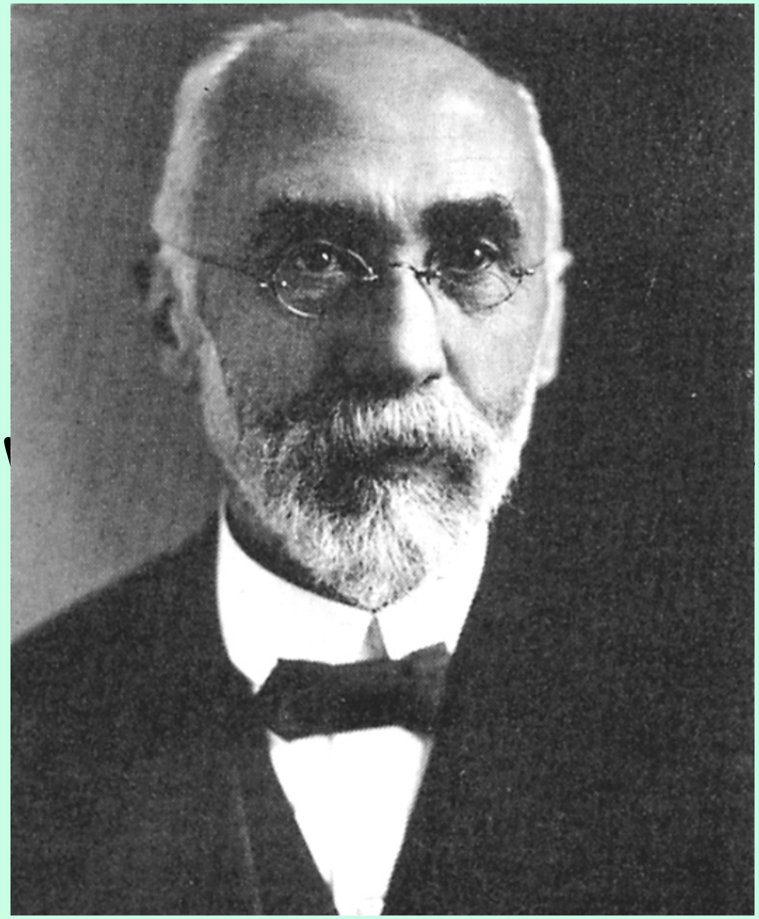
Other names should be mentioned in the preparation of the great revolution to come, such as Hendrik Antoon LORENTZ



Hendrik Antoon LORENTZ

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Exactly 100 years ago, in 1905, a big step was made by Albert EINSTEIN, who proposed that Space and Time were not absolute.

Two mathematicians played an important rôle in examining the mathematical background of this new physics, namely Henri POINCARÉ and Hermann MINKOWSKI.

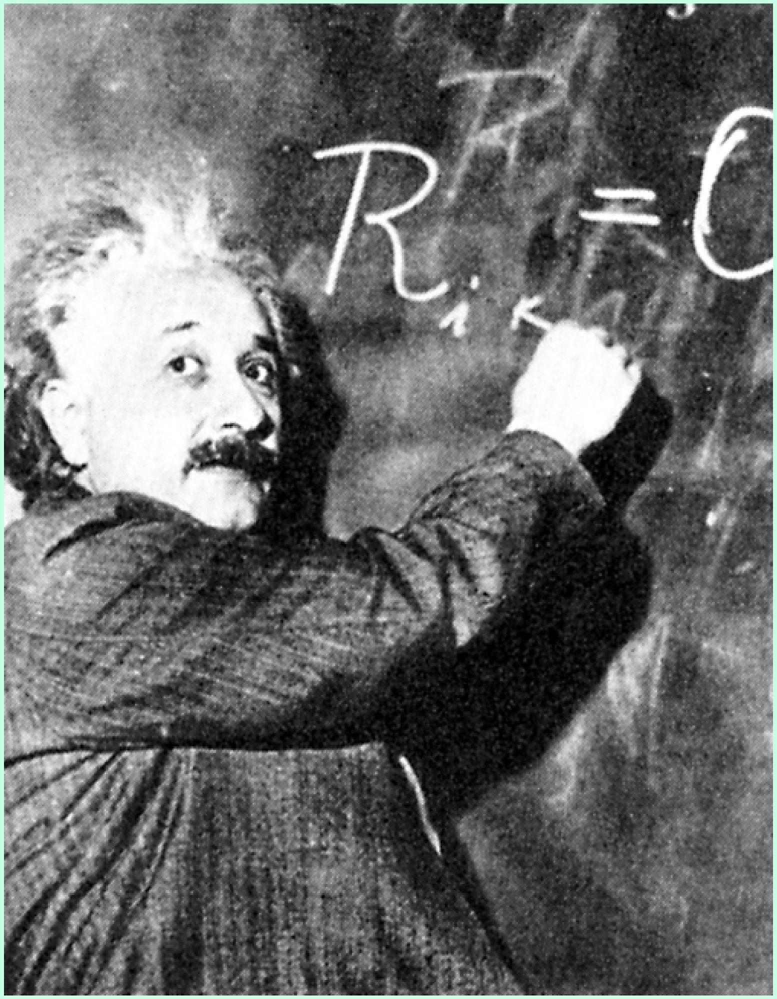
Hermann MINKOWSKI

Space-time is structured by a new type of geometrical structure that mathematicians had not studied previously in great detail, a so-called Lorentzian metric with signature (- + + +).

Some of the best exposition of the philosophical questions around the question of Space between Physics and Mathematics is Hermann WEYL

Hermann WEYL

As you know, Albert EINSTEIN went later one step further in his General Theory of Relativity, that revolutionised the Theory of Gravitation.



Albert EINSTEIN

As you know, Albert EINSTEIN went later one step further in his General Theory of Relativity, that revolutionised the theory of gravitation.

For that, he was greatly helped in his mathematical quest by his colleague at the ETH Zurich Marcel GROSSMANN

Marcel GROSSMANN

In this theory, the metric is determined by the position of the matter and other physical fields.

The field equations have been obtained through a competition-cooperation between HILBERT and EINSTEIN. They involve the form of the curvature introduced by RICCI for purely geometric reasons.

These new models are worthy of thorough investigations, such as the Schwarzschild metric that models black holes.

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These new models are worthy of thorough investigations, such as the Schwarzschild metric that models black holes.

VII. Modern Times

The broadening of Geometry went on in the first part of the century.

One of the great geometers of that time was Élie CARTAN

Élie CARTAN

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But many other contributions were made by several people such as Shiing Shen CHERN who developed many aspects of Geometry and their relations in his « *Geometry of G-structures* »

The second part of the century saw also the emergence of other aspects such as a Non-commutative Geometry by Alain CONNES.

Alain CONNES

The story leading to abundance of Geometries continues all along the XXth century.

To close this lecture, I single out one sequence that is remarkable in the way it developed:

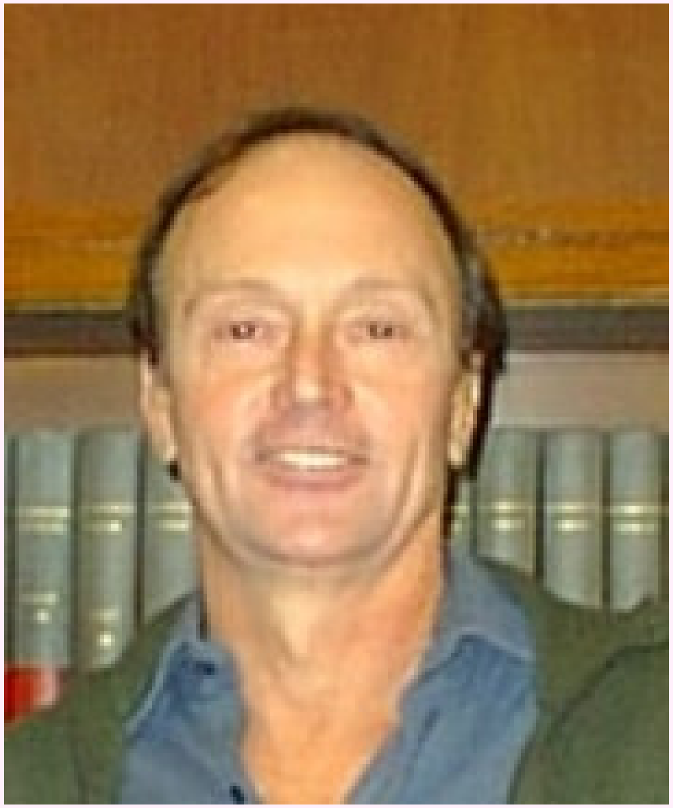
- ✔ *solving the Einstein equation turned out to be a very difficult problem;*

- ✔ *one possibility is to use the space of all Geometries that one can put on a space slice in space-time, a priori a monster;*

- ✔ *solving the Einstein equation reduces to evolving the metric induced on a space slice thanks to its Ricci curvature.*

The idea of studying systematically the deformation of the metric by the Ricci curvature was used for geometric purposes:

- ✔ *in the 1970s by Thierry AUBIN, JPB, …*
- ✔ *in a systematic way in the 1980s by Richard HAMILTON*



Richard HAMILTON

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- ✔ *in the 1970s by Thierry AUBIN, JPB, …*

- ✔ *in a systematic way in the 1980s by Richard HAMILTON who established that the Ricci flow defines a true dynamic among metrics;*

- ✔ *he obtained very important results in dimension 3, proving that the 3-sphere was the only (simply connected) space admitting positively curved metrics.*

At the turn of the century, Henri POINCARÉ introduced Topology, i.e. the study of spaces up to deformations:

- ✔ *he was able to associate an algebraic invariant to any space, its homology…*
- ✔ *and came across the problem of characterizing the 3-sphere because of its very simple homology…*
- ✔ *until he constructed a «fake» 3-sphere having the same homology.*

He then introduced a more refined invariant for a space, now called the Poincaré group (or the fundamental group) and proved

- ✔ *that the 3-sphere has a fundamental group reduced to one element, this means that every loop on the 3-sphere can be deformed to a point;*
- ✔ *and asked the question whether this was characteristic of the 3-sphere among closed spaces;*
- ✔ *this purely topological question, that resisted many attempts, was known later as the Poincaré Conjecture.*

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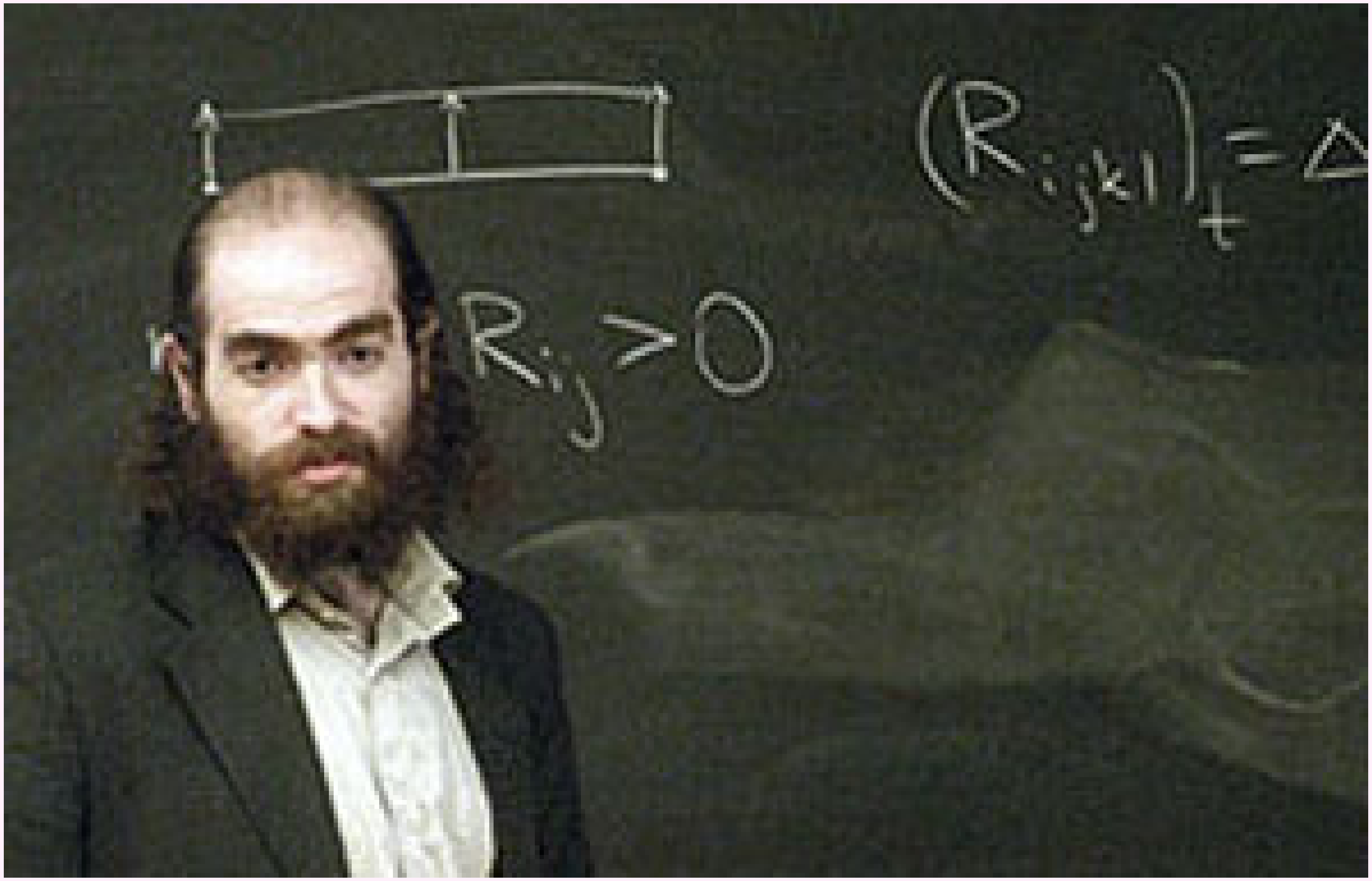

Grigory PERELMAN

Then the situation was turned around by Grigory PERELMAN who:

- ✔ *was able to show that, under no curvature condition, the Ricci flow could develop singularities that could be controlled;*
- ✔ *and proposed a way to derive from his work that, on a simply connected 3-space, the metric could converge to a constant curvature metric, hence solving the Poincaré Conjecture.*

This was a great success for Metric Geometry, proving the depth of the Ricci curvature.

Thank you

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