1 Background

In 1998, Diaconis and Holmes gave a bijection to the the set of perfect matchings $\mathcal{M}_n$ on $2n$ points (Partition of the $2n$ elements into n pairs) and the set of phylogenetic tree with $n-1$ leaves. The bijection works by considering all pair (a, b) as a pair of siblings from same parent, and systematically labeling the internal nodes.

With this token, we can say that a random walk on $\mathcal{M}_n$ shall correspond to a random walk on the phylogenetic trees.

There is a natural action of S_n on $\mathcal{M}_n$. If $w = \{(a_i, b_i) : \uplus(a_i, b_i) = \{1, ..., 2n\}\}$ is a perfect matching and $g \in S_{2n}$, then the action is defined by

$$g \cdot w = \{(a_{g(i), g(i)})\}$$

A step of a walk from $w \in \mathcal{M}_n$ is given by picking adjacent transposition σ uniformly from

$$\{\eta \in S_{2n}; \eta \cdot w \neq g\}$$

and moving to $\sigma \cdot g$. Starting from any state x, the unique stationary distribution to this walk is uniform. Marking that $|\mathcal{M}_n| = \frac{(2n)!}{2^n n!}$, $\pi(x) = \frac{2^n n!}{(2n)!}$ is the uniform distribution. Persi gives a sharp bound on the rate of convergence to the stationary distribution. This problem is close to the one dealing with the number of shuffles required to 'almost' achieve uniform randomness in the deck of cards. Scarabotti and Diaconis had dealt with this problem using the technique of Gelfand pairs and Sphecical functions from the theory of Representation theory on S_n.

2 Results

Persi considers the specific random walk on the perfect matchings with the following transition matrix:

$$K(x, y) = \begin{cases} \frac{1}{n(n-1)} & \text{if } \sigma x = y \text{ for some transposition } \sigma \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

A useful tool in analysing the convergence speed to the stationary distribution π will be a spectral decomposition. In particular, if $F = [f_0, f_1, ...f_n]$ is a matrix of orthonormal eigen vectors with eigenvalues b_i , $f_0 = \pi, \quad b_0 = 1$ and

$$\begin{aligned} \|K_x(\cdot)^m - \pi(\cdot)\|_2^2 &= (K_x^m - \pi)I(K_x^m - \pi)^T \\ &= (e_x K^m - \pi)F[F(e_x K^m - \pi)]^T \\ &= \|[(b_i^m f_i(x) - \pi \cdot f_i)]_i\|_2 \\ &= \|[(b_i^m f_i(x)) - b_0 f_0(x)^2]\|_2 \\ &= \sum_{i \neq 0} b_i^{2m} f_i^2(x) \end{aligned} \tag{2.2}$$

If a group G acts transitively on $\mathcal{X}$ and $K(\sigma(x), \sigma(y)) = K(x, y)$, then $\|K_x - \pi\|_2$ should not depend on the x. In that case, adding over x and using the orthonormality, we see

$$\|K_x(\cdot)^m - \pi(\cdot)\|_2^2 = \frac{1}{|\mathcal{M}|} \sum_{i \neq 0} b_i^{2m} \sum_x f_i^2(x) = \frac{1}{|\mathcal{M}|} \sum_{i \neq 0} b_i^{2m}$$

Clearly, the group S_{2n} acts on the set of perfect pairing, and its action is dictated by the orbits. If B_n is the stabilizer of the identity matching

$$(1,2)(3,4)\cdots(2n-1,2n)$$

Then B_n is isomorphic to $S_2 \; wr_{\{1,2,\ldots,n\}} \; S_n$, or hyperoctahedral group (size $2^n n!$). We therefore identify all matchings to the cosets of B_n in S_n. Call $\mathcal{M} = S_n / B_n$. Then $\mathbb{R}\mathcal{M}$ (real functions on $\mathcal{M}$) can be decomposed into representations of S_{2n}s. This is well studied. For each $\lambda \vdash n$, there exists spect modules $\mathcal{S}^{2\lambda}$ such that

$$\mathcal{L}(\mathcal{M}_n) = \bigoplus_{\lambda \vdash n} \mathcal{S}^{2\lambda_k}.$$

This prompts us to express K as a representation of S_n, and this can be indeed be done.

Proposition 2.1. *If D is a regular representation of S_n, then*

$$K = \frac{2n-1}{2n-2}\left(D(T_n) - \frac{1}{2n-1} I\right)$$

If $T_n = \frac{1}{\binom{2n}{n}} \sum (i, j)$.

Proof. $D(T_n)$ is about casting all transpositions to each mactchings and averaging the answer. For each matching ρ, $2n/\binom{2n}{2} = \frac{1}{2n-1}$ transpositions fixes ρ. We can get K by removing the diagonal $\frac{1}{2n-1}$ in $D(T_n)$ and normalizing it. □

The idea is to cast Fourier Transform on T_n about each irreducible representations appeaching in $\mathbb{R}\mathcal{M}_n$. Since T_n is a module homomorphism from irreducible to irreducible, Schur's lemma helps us to extract the information about the eigenvalues b_i of T_n, and hence that of K. Based on the information about $S^{2\lambda}$, these b_is can be computed. After the computation, we get the following result.

Theorem 2.2. *For the Markov chain K on $\mathcal{M}_n$, then for any x, if $m = \frac{1}{2}(\log n + c)$ with $c > 0$,*

$$\|K_x^m - \pi\| \leq ae^{-c}$$

The result is sharp in that, if $m = \frac{1}{2}(\log n - c)$, there is positive ϵ such that

$$\|K_x^m - \pi\| \geq \epsilon$$

for all n.

Thus, we need $m = \frac{1}{2}(\log n + c)$ number of shuffling in the phylogenetic tree to achieve uniform distribution.