RECONSTRUCTION ON TREES: BEATING THE SECOND EIGENVALUE

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Reference: [1] [2].

1. Introduction

We consider the following broadcasting process. The first building block of the process is an irreducible aperiodic Markov chain (or channel) on a finite alphabet $\mathscr{A} = \{1, \ldots, k\}$. We will denote by $\mathbf{M}_{i,\,j}$ the transition probability from i to j by $\mathbf{P}[M(i) = j] = \mathbf{M}_{i,\,j}$; and by $\lambda_2(M)$ the eigenvalue of $\mathbf{M}$ which has the second largest absolute value ($\lambda_2(M)$ may be negative). The second building block is a d-ary tree $T = T_d = (V_d, E_d)$ rooted at ρ. At the root ρ, one of the symbols of $\mathscr{A}$ is chosen according to an initial distribution $\pi = (\pi_1, \ldots, \pi_k)$. We denote this (random) symbol by σ_ρ. This symbol is then propagated in the tree in the following way. For each vertex v having as a parent v', we let $\sigma_v = M_{v',\,v}(\sigma_{v'})$, where the $M_{v',\,v}$ are independent copies of M. Equivalently, for a vertex v, let v' be the parent of v, and let A_v be the set of all vertices which are connected to ρ through paths which do not contain v. Then we have

$$\mathbf{P}[\sigma_v = j|(\sigma_\omega)_{\omega\in A_v}] = \mathbf{P}[\sigma_v = j|\sigma_{v'}] = \mathbf{M}_{\sigma_{v'},\,j}.$$

It is known that the reconstruction of T is possible if $d\lambda_2^2(M) > 1$. Moreover, in this case it is possible to reconstruct using a majority algorithm which ignores the location of the data at the boundary of the tree. In this paper we show that, both for the binary asymmetric channel and for the symmetric channel on many symbols, it is sometimes possible to reconstruct even when $d\lambda_2^2(M) < 1$. This result indicates that, for many (maybe most) tree-indexed Markov chains, the location of the data on the boundary plays a crucial role in reconstruction problems.

2. Definitions and Background

Let $d(,)$ denote the graph-metric distance on T, and let $L_n = \{v \in V : d(\rho, v) = n\}$ be the nth level of the tree. We denote by $\sigma_{L_n} = (\sigma(v))_{v\in L_n}$ the symbols at the nth level of the tree. We let $c_{L_n} = (c_{L_n}(1), \ldots, c_{L_n}(k))$, where

$$c_{L_n}(i) = \#\{v \in L_n : \sigma(v) = i\}.$$

That is, c_{L_n} is the count of the nth level. Note that both $(\sigma_{L_n})_{n=1}^{\infty}$ and $(c_{L_n})_{n=1}^{\infty}$ are Markov chains. We want to know if the data on the boundary gives some information on the root.

Definition 1. We say that the reconstruction problem is **solvable** if there exists $i, j \in \mathscr{A}$ for which

$$\lim_{n\to\infty} |\mathbf{P}_n^i - \mathbf{P}_n^j| > 0,$$

where $|\ |$ denote the total variance norm and $\mathbf{P}_n^l$ denote the conditional distribution of σ_{L_n} given that $\sigma_\rho = l$.

Definition 2. We say that the reconstruction problem is **count-solvable** if there exists $i, j \in \mathscr{A}$ for which

$$\lim_{n\to\infty} |\mathbf{P}_n^{(c),\, i} - \mathbf{P}_n^{(c),\, j}| > 0,$$

where $|\ |$ denote the total variance norm and $\mathbf{P}_n^{(c),\, l}$ denote the conditional distribution of c_{L_n} given that $\sigma_\rho = l$.

Definition 3. Let T' be a subtree of the tree T which is rooted at ρ. We say that T' is an **l-diluted b-regular tree** if, for all i, all the vertices of T' at level il have exactly b descendents at level $(i+1)l$.

3. Main Results

Theorem 1. *Consider the asymmetric binary chains*

$$\mathbf{M}_1 = \begin{pmatrix} 1-\delta_1 & \delta_1 \\ 1-\delta_2 & \delta_2 \end{pmatrix}.$$

[*Note that* $\lambda_2(M_1) = \delta_2 - \delta_1$]. *Suppose that* $0 \le \lambda \le 1$ *and that* $d\lambda > 1$*; then there exists a* $\delta > 0$ *s.t. if* $\lambda_2(M_1) = \lambda$ *and* $\delta_1 < \delta$*, then the reconstruction problem is solvable for the d-ary tree and the chain* M_1.

Theorem 2. *Consider the symmetric chains on q symbols:*

$$\mathbf{M}_2 = \begin{pmatrix} 1-(q-1)\delta & \delta & \dots & \delta \\ \delta & 1-(q-1)\delta & \delta & \dots \\ \vdots & \dots & \ddots & \vdots \\ \delta & \dots & \delta & 1-(q-1)\delta \end{pmatrix}.$$

[Note that $\lambda_2(M_2) = 1 - q\delta$]. Let $0 < \delta < 1$ and take d such that $d\lambda > 1$. Then there exists a Q s.t. if $q > Q$ and $\lambda = 1 - q\delta$, then the reconstruction problem is solvable for the d-ary tree and the chain M_2.

Theorems 1 and 2 are sharp in the following sense.

Proposition 3. *Take d an integer s.t. $|d\lambda_2(M_1)| \leq 1$. Then the reconstruction problem is unsolvable for the d-ary tree and M_1.*

Proposition 4. *Let $\lambda = 1 - q\delta$. Suppose that $0 \leq d\lambda \leq 1$. Then the reconstruction problem is unsolvable for the d-ary tree and the chain M_2.*

Proposition 5. *There exists a channel M such that $\lambda_2(M) = 0$ and such that the reconstruction problem is solvable for M and all $d \geq 1000$.*

4. Proof Outline

The proofs of Theorems 1 and 2 and of Propositions 3 and 4 all use random-cluster arguments. Consider the space $\{0,1\}^{E_d}$, an element of which is $(\tau(e))_{e \in E_d}$. By $\lambda - percolation$ on T we mean the random process which has the state space $\{0,1\}^{E_d}$ and for which $\mathbf{P}[\tau(e) = 1] = \lambda$ independently for all $e \in E_d$. An edge e with $\tau(e) = 1$ is called an open edge. More generally, we say that a subtree $T' = (V', E')$ of T is open if all the edges $e \in E'$ are open. The following lemmas play a key role in our proof.

Lemma 6. *Let T_d be the infinite rooted d-ary tree, and let $0 \leq \lambda \leq 1$ be a number such that $d\lambda > 1$. There exists a positive $\epsilon = \epsilon(d,\lambda)$ s.t. for all $b \geq 1$, there exists $l \geq 1$ s.t. if one performs percolation with parameter $\lambda' \geq \lambda$ on T, then*

$$\mathbf{P}[\rho\ is\ the\ root\ of\ an\ open\ l - diluted\ b - regular\ tree] \geq \epsilon(d,\lambda).$$

We also need a complementary result for λ close to 1.

Lemma 7. *Let T_d be the infinite rooted d-ary tree, and take $l \geq 1$ and $\epsilon > 0$. There exists $\lambda < 1$ such that, if one performs percolation with parameter $\lambda' \geq \lambda$ on T, then*

$$\mathbf{P}[\rho\ is\ the\ root\ of\ an\ open\ l - diluted\ (d^l - 1) - regular\ tree] \geq 1 - \epsilon.$$

The proof of Propositions 3 and 4 uses another type of random-cluster argument. The channels for which we can use this kind of argument are channels M which have matrices $(\mathbf{M}_{i,\ j})_{i,\ j=1}^{k}$ which satisfy

$$\mathbf{M}_{i,\ j} = \lambda \mathbf{N}_{i,\ j} + (1-\lambda)\nu_j \qquad (*)$$

for some channel N which has the matrix $(\mathbf{N}_{i,\ j})_{i,\ j=1}^{k}$, a distribution vector $(\nu_j)_{j=1}^{k}$ and a number $0 \leq \lambda \leq 1$.

Proposition 8. *Suppose that M has the form $(*)$. Then the reconstruction problem for M is unsolvable whenever $d\lambda \leq 1$.*

Proposition 9. *All binary channels M in Theorem 1 have the form $(*)$ with $\lambda = \lambda_2(M)$. All symmetric channels M in Theorem 2 with $\lambda = 1-q\delta \geq 0$ have the form $(*)$ with $\lambda = \lambda_2(M) = 1 - q\delta$.*

Reference

[1] Evans, W., Kenyon, C., Peres, Y. and Schulman L. J. (2000). *Broadcasting on trees and the Ising model*. Ann. Appl. Probab. 10 410C433.

[2] Kesten, H. and Stigum, B. P. (1966). *Additional limit theorem for indecomposable multidimensional GaltonCWatson processes*. Ann. Math. Statist. 37 1463C1481.