Lecture 7 : Martingales bounded in L^2

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapters 0, 12], [Dur10, Section 4.4], [AN72, Section I.6].

1 Preliminaries

DEF 7.1 *For $1 \leq p < +\infty$, we say that $X \in \mathcal{L}^p$ if*

$$\|X\|_p = \mathbb{E}[|X^p|]^{1/p} < +\infty.$$

By Jensen's inequality, for $1 \leq p \leq r < +\infty$ we have $\|X\|_p \leq \|X\|_r$ if $X \in \mathcal{L}^r$.

Proof: For $n \geq 0$, let

$$X_n = (|X| \wedge n)^p.$$

Take $c(x) = x^{r/p}$ on $(0, +\infty)$ which is convex. Then

$$(\mathbb{E}[X_n])^{r/p} \leq \mathbb{E}[(X_n)^{r/p}] = \mathbb{E}[(|X| \wedge n)^r] \leq \mathbb{E}[|X|^r].$$

Take $n \to \infty$ and use (MON). ■

DEF 7.2 *We say that X_n converges to X_∞ in $\mathcal{L}^p$ if $\|X_n - X_\infty\|_p \to 0$. By the previous result, convergence on $\mathcal{L}^r$ implies convergence in $\mathcal{L}^p$ for $r \geq p \geq 1$.*

LEM 7.3 *Assume $X_n, X_\infty \in \mathcal{L}^1$. Then*

$$\|X_n - X_\infty\|_1 \to 0,$$

implies

$$\mathbb{E}[X_n] \to \mathbb{E}[X_\infty].$$

Proof: Note that

$$|\mathbb{E}[X_n] - \mathbb{E}[X_\infty]| \leq \mathbb{E}|X_n - X_\infty| \to 0.$$

■

DEF 7.4 *We say that $\{X_n\}_n$ is* bounded in $\mathcal{L}^p$ *if*

$$\sup_n \|X_n\|_p < +\infty.$$

2 L^2 convergence

THM 7.5 *Let M be a MG with $M_n \in \mathcal{L}^2$. Then M is bounded in $\mathcal{L}^2$ if and only if*

$$\sum_{k\geq 1} \mathbb{E}[(M_k - M_{k-1})^2] < +\infty.$$

When this is the case, M_n converges a.s. and in $\mathcal{L}^2$.

Proof:

LEM 7.6 (Orthogonality of increments) *Let $s \leq t \leq u \leq v$. Then,*

$$\langle M_t - M_s, M_v - M_u \rangle = 0.$$

Proof: Use $M_u = \mathbb{E}[M_v \,|\, \mathcal{F}_u]$, $M_t - M_s \in \mathcal{F}_u$ and apply the L^2 characterization of conditional expectations. ■

That implies

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{1\leq i\leq n} \mathbb{E}[(M_i - M_{i-1})^2],$$

proving the first claim.

By monotonicity of norms, M is bounded in L^2 implies M bounded in L^1 which, in turn, implies M converges a.s. Then using (FATOU) in

$$\mathbb{E}[(M_{n+k} - M_n)^2] = \sum_{n+1\leq i\leq n+k} \mathbb{E}[(M_i - M_{i-1})^2],$$

gives

$$\mathbb{E}[(M_\infty - M_n)^2] \leq \sum_{n+1\leq i} \mathbb{E}[(M_i - M_{i-1})^2].$$

The RHS goes to 0 which proves the second claim.

■

3 Back to branching processes

THM 7.7 *Let Z be a branching process with $Z_0 = 1$, $m = \mathbb{E}[X(1,1)] > 1$ and $\sigma^2 = \mathrm{Var}[X(1,1)] < +\infty$. Then, $M_n = m^{-n} Z_n$ converges in L^2, and in particular, $\mathbb{E}[M_\infty] = 1$.*

Proof: From the orthogonality of increments

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_{n-1}^2] + \mathbb{E}[(M_n - M_{n-1})^2].$$

On $\{Z_{n-1} = k\}$

$$\begin{aligned}
\mathbb{E}[(M_n - M_{n-1})^2 \,|\, \mathcal{F}_{n-1}] &= m^{-2n}\mathbb{E}[(Z_n - mZ_{n-1})^2 \,|\, \mathcal{F}_{n-1}] \\
&= m^{-2n}\mathbb{E}[(\sum_{i=1}^{k} X(i,n) - mk)^2 \,|\, \mathcal{F}_{n-1}] \\
&= m^{-2n}k\sigma^2 \\
&= m^{-2n}Z_{n-1}\sigma^2.
\end{aligned}$$

Hence

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_{n-1}^2] + m^{-n-1}\sigma^2.$$

Since $\mathbb{E}[M_0^2] = 1$,

$$\mathbb{E}[M_n^2] = 1 + \sigma^2 \sum_{i=2}^{n+1} m^{-i},$$

which is uniformly bounded when $m > 1$. So M_n converges in L^2. Finally by (FATOU)

$$\mathbb{E}|M_\infty| \leq \sup \|M_n\|_1 \leq \sup \|M_n\|_2 < +\infty$$

and

$$|\mathbb{E}[M_n] - \mathbb{E}[M_\infty]| \leq \|M_n - M_\infty\|_1 \leq \|M_n - M_\infty\|_2,$$

implies the convergence of expectations. ■

In a homework problem, we will show that under the assumptions of the previous theorem

$$\{M_\infty = 0\} = \{Z_n = 0, \text{ for some } n\},$$

and

$$\mathbb{P}[M_\infty = 0] = \pi,$$

the probability of extinction.

EX 7.8 (Geometric Offspring) *Assume*

$$0 < p < 1,\ q = 1 - p,\ p_i = pq^i,\ \forall i \geq 0,\ m = \frac{q}{p}.$$

Then

$$f(s) = \frac{p}{1 - sq},\ \pi = \min\{\frac{p}{q}, 1\}.$$

- **Case $m \neq 1$.** *If G is a 2×2 matrix, denote*

$$G(s) = \frac{G_{11}s + G_{12}}{G_{21}s + G_{22}}.$$

Then $G(H(s)) = (GH)(s)$. By diagonalization,

$$\begin{pmatrix} 0 & p \\ -q & 1 \end{pmatrix}^n = (q-p)^{-1} \begin{pmatrix} 1 & p \\ 1 & q \end{pmatrix} \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \begin{pmatrix} q & -p \\ -1 & 1 \end{pmatrix}$$

leading to

$$f_n(s) = \frac{pm^n(1-s) + qs - p}{qm^n(1-s) + qs - p}.$$

In particular, when $m < 1$ we have $\pi = \lim f_n(0) = 1$. On the other hand, if $m > 1$, we have by (DOM) for $\lambda \geq 0$

$$\begin{aligned} \mathbb{E}[\exp(-\lambda M_\infty)] &= \lim_n f_n(\exp(-\lambda/m^n)) \\ &= \frac{p\lambda + q - p}{q\lambda + q - p} \\ &= \pi + (1-\pi)\frac{(1-\pi)}{\lambda + (1-\pi)}. \end{aligned}$$

The first term corresponds to a point mass at 0 and the second term corresponds to an exponential with mean $1/(1-\pi)$.

- **Case $m = 1$.** *By induction*

$$f_n(s) = \frac{n - (n-1)s}{n + 1 - ns},$$

so that

$$\mathbb{P}[Z_n > 0] = 1 - f_n(0) = \frac{1}{n+1},$$

and

$$\mathbb{E}[e^{-\lambda Z_n/n} \mid Z_n > 0] = \frac{f_n(e^{-\lambda/n}) - f_n(0)}{1 - f_n(0)} \to \frac{1}{1+\lambda},$$

which is the Laplace transform of an eponential mean 1. This is consistent with $\mathbb{E}[Z_n] = 1$.

References

[AN72] Krishna B. Athreya and Peter E. Ney. *Branching processes*. Springer-Verlag, New York, 1972.

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.