Lecture 22 : Strong Markov property

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Dur10, Section 8.3], [Lig10, Section 1.8], [MP10, Section 2.2].

1 Stopping times

We first generalize stopping times to continuous time.

DEF 22.1 (Stopping time) *A RV T with values in $[0,+\infty]$ is a stopping time with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if for all $t \geq 0$,*

$$\{T \leq t\} \in \mathcal{F}(t).$$

THM 22.2 *If the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is right-continuous in the previous definition, then an equivalent definition is obtained by using a strict inequality.*

EX 22.3 *Let G be an open set. Then*

$$T = \inf\{t \geq 0 \,:\, B(t) \in G\},$$

is a stopping time with respect to $\{\mathcal{F}^+(t)\}$. Indeed, note

$$\{T < t\} = \bigcup_{s<t,\ s\in\mathbb{Q}} \{B(s) \in G\} \in \mathcal{F}^+(t),$$

by continuity of paths and the fact that G is open.

To define the strong Markov property, we will need the following.

DEF 22.4 *Let T be a stopping time with respect to $\{\mathcal{F}^+(t)\}_{t\geq 0}$. Then we let*

$$\mathcal{F}^+(T) = \{A \,:\, A \cap \{T \leq t\} \in \mathcal{F}^+(t),\ \forall t \geq 0\}.$$

The following lemma will be useful in extending properties about discrete-time stopping times to continuous time.

LEM 22.5 *The following hold:*

1. *If T_n is a sequence of stopping times with respect to $\{\mathcal{F}(t)\}$ such that $T_n \uparrow T$, then so is T.*
2. *Let T be a stopping time with respect to $\{\mathcal{F}(t)\}$. Then the following are also stopping times:*

$$T_n = (m+1)2^{-n} \text{ if } m2^{-n} \leq T < (m+1)2^{-n}.$$

EX 22.6 *Let F be a closed set. Then*

$$T = \inf\{t \geq 0 \,:\, B(t) \in F\},$$

is a stopping time with respect to $\{\mathcal{F}^+(t)\}$. See [Lig10] for a proof.

2 Strong Markov property

THM 22.7 (Strong Markov property) *Let $\{B(t)\}_{t\geq 0}$ be a BM and T, an almost surely finite stopping time. Then the process*

$$\{B(T+t) - B(T) \,:\, t \geq 0\},$$

is a BM started at 0 independent of $\mathcal{F}^+(T)$.

Proof: Let T_n be a discretization of T as above. Let

$$B_k(t) = B(t + k2^{-n}) - B(k2^{-n}),$$

and

$$B_*(t) = B(t + T_n) - B(T_n).$$

Suppose $E \in \mathcal{F}^+(T_n)$. Then for every "finite-dimensional" event A we have, by the Markov property and time translation invariance,

$$\begin{aligned}
\mathbb{P}[\{B_* \in A\} \cap E] &= \sum_{k=1}^{+\infty} \mathbb{P}[\{B_k \in A\} \cap E \cap \{T_n = k2^{-n}\}] \\
&= \sum_{k=1}^{+\infty} \mathbb{P}[B_k \in A]\mathbb{P}[E \cap \{T_n = k2^{-n}\}] \\
&= \mathbb{P}[B \in A] \sum_{k=1}^{+\infty} \mathbb{P}[E \cap \{T_n = k2^{-n}\}] \\
&= \mathbb{P}[B \in A]\mathbb{P}[E].
\end{aligned}$$

That is, B_* is independent of $\mathcal{F}^+(T_n)$. Since $\mathcal{F}^+(T) \subseteq \mathcal{F}^+(T_n)$, B_* is also independent of $\mathcal{F}^+(T)$. Moreover, $T_n \downarrow T$ so that by continuity $\{B(t+T) - B(T)\}_{t\geq 0}$ is itself independent of $\mathcal{F}^+(T)$. The same argument shows that the increments have the correct distribution. ■

3 Applications

We discuss one application.

THM 22.8 (Reflection principle) *Let $\{B(t)\}_{t\geq 0}$ be a standard BM and T, a stopping time. Then the process*

$$B^*(t) = B(t)\mathbb{1}\{t \leq T\} + (2B(T) - B(t))\mathbb{1}\{t > T\},$$

called BM reflected at T, *is also a standard BM.*

Proof: Follows immediately from the strong Markov property and symmetry. ■

A remarkable consequence is the following.

THM 22.9 *Let $\{B(t)\}$ be a standard BM and let*

$$M(t) = \max_{0\leq s\leq t} B(s).$$

Then, if $a > 0$,

$$\mathbb{P}[M(t) \geq a] = 2\mathbb{P}[B(t) \geq a] = \mathbb{P}[|B(t)| \geq a].$$

Proof: Let

$$T = \inf\{t \geq 0 \,:\, B(t) = a\}.$$

Then we have the disjoint union

$$\begin{aligned}
\{M(t) \geq a\} &= \{B(t) \geq a\} \cup \{B(t) < a, M(t) \geq a\} \\
&= \{B(t) \geq a\} \cup \{B^*(t) > a\}.
\end{aligned}$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Lig10] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.