Lecture 11 : UI MGs

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 14], [Dur10, Section 4.5].

1 UI MGs

THM 11.1 (Convergence of UI MGs) *Let M be UI MG. Then*

$$M_n \to M_\infty,$$

a.s. and in L^1. Moreover,

$$M_n = \mathbb{E}[M_\infty \,|\, \mathcal{F}_n], \qquad \forall n.$$

Proof: UI implies L^1-bddness so we have $M_n \to M_\infty$ a.s. By necessary and sufficient condition, we also have L^1 convergence.

Now note that for all $r \geq n$ and $F \in \mathcal{F}_n$, we know $\mathbb{E}[M_r \,|\, \mathcal{F}_n] = M_n$ or

$$\mathbb{E}[M_r; F] = \mathbb{E}[M_n; F],$$

by definition of CE. We can take a limit by L^1 convergence. More precisely

$$|\mathbb{E}[M_r; F] - \mathbb{E}[M_\infty; F]| \leq \mathbb{E}[|M_r - M_\infty|; F] \leq \mathbb{E}[|M_r - M_\infty|] \to 0,$$

as $r \to \infty$. So plugging above

$$\mathbb{E}[M_\infty; F] = \mathbb{E}[M_n; F],$$

and $\mathbb{E}[M_\infty \,|\, \mathcal{F}_n] = M_n$. ■

2 Applications I

THM 11.2 (Levy's upward thm) *Let $Z \in L^1$ and define $M_n = \mathbb{E}[Z \,|\, \mathcal{F}_n]$. Then M is a UI MG and*

$$M_n \to M_\infty = \mathbb{E}[Z \,|\, \mathcal{F}_\infty],$$

a.s. and in L^1.

Proof: M is a MG by (TOWER). We first show it is UI:

LEM 11.3 *Let* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. *Then*

$$\{\mathbb{E}[X \,|\, \mathcal{G}] \,:\, \mathcal{G} \text{ is a sub-}\sigma\text{-field of } \mathcal{F}\},$$

is UI.

Proof: We use the absolute continuity lemma again. Let $Y = \mathbb{E}[X \,|\, \mathcal{G}] \in \mathcal{G}$. Since $\{|Y| > K\} \in \mathcal{G}$,

$$\begin{aligned}
\mathbb{E}[|Y|; |Y| > K] &= \mathbb{E}[|\mathbb{E}[X \,|\, \mathcal{G}]|; |Y| > K] \\
&\leq \mathbb{E}[\mathbb{E}[|X| \,|\, \mathcal{G}]; |Y| > K] \\
&= \mathbb{E}[|X|; |Y| > K].
\end{aligned}$$

By Markov

$$\mathbb{P}[|Y| > K] \leq \frac{\mathbb{E}|Y|}{K} \leq \frac{\mathbb{E}|X|}{K} \leq \delta,$$

for K large enough (uniformly in $\mathcal{G}$). And we are done. ■

In particular, we have convergence a.s. and in L^1 to $M_\infty \in \mathcal{F}_\infty$.

Let $Y = \mathbb{E}[Z \,|\, \mathcal{F}_\infty] \in \mathcal{F}_\infty$. By dividing into negative and positive parts, we assume $Z \geq 0$. We want to show, for $F \in \mathcal{F}_\infty$,

$$\mathbb{E}[Z; F] = \mathbb{E}[M_\infty; F].$$

By Uniqueness Lemma, it suffices to prove equality for all $\mathcal{F}_n$. If $F \in \mathcal{F}_n \subseteq \mathcal{F}_\infty$, then by (TOWER)

$$\mathbb{E}[Z; F] = \mathbb{E}[Y; F] = \mathbb{E}[M_n; F] = \mathbb{E}[M_\infty; F].$$

■

THM 11.4 (Levy's $0-1$ law) *Let* $A \in \mathcal{F}_\infty$. *Then*

$$\mathbb{P}[A \,|\, \mathcal{F}_n] \to \mathbb{1}_A.$$

Proof: Immediate. ■

COR 11.5 (Kolmogorov's $0-1$ law) *Let* $X_1, X_2, \ldots$ *be iid RVs. Recall that the tail* σ*-field is*

$$\mathcal{T} = \cap_n \mathcal{T}_n = \cap_n \sigma(X_{n+1}, X_{n+2}, \ldots).$$

If $A \in \mathcal{T}$ *then* $\mathbb{P}[A] \in \{0, 1\}$.

Proof: Since $A \in \mathcal{T}_n$ is independent of $\mathcal{F}_n$,

$$\mathbb{P}[A \,|\, \mathcal{F}_n] = \mathbb{P}[A],$$

$\forall n$. By Levy's law,

$$\mathbb{P}[A] = \mathbb{1}_A \in \{0, 1\}.$$

■

3 Applications II

THM 11.6 (Levy's Downward Thm) *Let $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{G}_{-n}\}_{n \geq 0}$ a collection of σ-fields s.t.*

$$\mathcal{G}_{-\infty} = \cap_k \mathcal{G}_{-k} \subseteq \cdots \subseteq \mathcal{G}_{-n} \subseteq \cdots \subseteq \mathcal{G}_{-1} \subseteq \mathcal{F}.$$

Define

$$M_{-n} = \mathbb{E}[Z \,|\, \mathcal{G}_{-n}].$$

Then

$$M_{-n} \to M_{-\infty} = \mathbb{E}[Z \,|\, \mathcal{G}_{-\infty}]$$

a.s. and in L^1.

Proof: We apply the same argument as in the Martingale Convergence Thm. Let $\alpha < \beta \in \mathbb{Q}$ and

$$\Lambda_{\alpha,\beta} = \{\omega \,:\, \liminf X_{-n} < \alpha < \beta < \limsup X_{-n}\}.$$

Note that

$$\begin{aligned}
\Lambda &\equiv \{\omega \,:\, X_n \text{ does not converge}\} \\
&= \{\omega \,:\, \liminf X_{-n} < \limsup X_{-n}\} \\
&= \cup_{\alpha < \beta \in \mathbb{Q}} \Lambda_{\alpha,\beta}.
\end{aligned}$$

Let $U_N[\alpha, \beta]$ be the *number of upcrossings of* $[\alpha, \beta]$ *between time* $-N$ *and* -1. Then by the Upcrossing Lemma applied to the MG $M_{-N}, \ldots, M_{-1}$

$$(\beta - \alpha)\mathbb{E}U_N[\alpha, \beta] \leq |\alpha| + \mathbb{E}|M_{-1}| \leq |\alpha| + \mathbb{E}|Z|.$$

By (MON)

$$U_N[\alpha, \beta] \uparrow U_\infty[\alpha, \beta],$$

and

$$(\beta - \alpha)\mathbb{E}U_\infty[\alpha, \beta] \leq |\alpha| + \mathbb{E}|Z| < +\infty,$$

so that

$$\mathbb{P}[U_\infty[\alpha, \beta] = \infty] = 0.$$

Since

$$\Lambda_{\alpha,\beta} \subseteq \{U_\infty[\alpha, \beta] = \infty\},$$

we have $\mathbb{P}[\Lambda_{\alpha,\beta}] = 0$. By countability, $\mathbb{P}[\Lambda] = 0$. Therefore we have convergence a.s.

By lemma in previous class, M is UI and hence we have L^1 convergence as well.

Finally, for all $G \in \mathcal{G}_{-\infty} \subseteq \mathcal{G}_{-n}$,

$$\mathbb{E}[Z; G] = \mathbb{E}[M_{-n}; G].$$

Take the limit $n \to +\infty$ and use L^1 convergence. ■

An application:

THM 11.7 (Strong Law; Martingale Proof) *Let $X_1, X_2, \ldots$ be iid RVs with $\mathbb{E}[X_1] = \mu$ and $\mathbb{E}|X_1| < +\infty$. Let $S_n = \sum_{i \leq n} X_n$. Then*

$$n^{-1} S_n \to \mu,$$

a.s. and in L^1.

Proof: Let

$$\mathcal{G}_{-n} = \sigma(S_n, S_{n+1}, S_{n+2}, \ldots) = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots),$$

and note that, for $1 \leq i \leq n$,

$$\mathbb{E}[X_1 \,|\, \mathcal{G}_{-n}] = \mathbb{E}[X_1 \,|\, S_n] = \mathbb{E}[X_i \,|\, S_n] = \mathbb{E}[n^{-1} S_n \,|\, S_n] = n^{-1} S_n,$$

by symmetry. By Levy's Downward Thm

$$n^{-1} S_n \to \mathbb{E}[X_1 \,|\, \mathcal{G}_{-\infty}],$$

a.s. and in L^1. Note that $\mathcal{G}_{-n} \subseteq \mathcal{E}_n$ and $\mathcal{G}_{-\infty} \subseteq \mathcal{E}$ so that $\mathcal{G}_{-\infty}$ is trivial and we must have $\mathbb{E}[X_1 \,|\, \mathcal{G}_{-\infty}] = \mu$. ■

4 Further material

DEF 11.8 *Let $X_1, X_2, \ldots$ be iid RVs. Let $\mathcal{E}_n$ be the σ-field generated by events invariant under permutations of the Xs that leave $X_{n+1}, X_{n+2}, \ldots$ unchanged. The* exchangeable σ-field *is $\mathcal{E} = \cap_m \mathcal{E}_m$.*

THM 11.9 (Hewitt-Savage 0-1 law) *Let $X_1, X_2, \ldots$ be iid RVs. If $A \in \mathcal{E}$ then $\mathbb{P}[A] \in \{0, 1\}$.*

Proof: The idea of the proof is to show that A is independent of itself. Indeed, we then have

$$0 = \mathbb{P}[A] - \mathbb{P}[A \cap A] = \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[A] = \mathbb{P}[A](1 - \mathbb{P}[A]).$$

Since $A \in \mathcal{E}$ and $A \in \mathcal{F}_\infty$, it suffices to show that $\mathcal{E}$ is independent of $\mathcal{F}_n$ for every n (by the π-λ theorem).

WTS: for every bounded ϕ, $B \in \mathcal{E}$,

$$\mathbb{E}[\phi(X_1, \ldots, X_k); B] = \mathbb{E}[\phi(X_1, \ldots, X_k)]\mathbb{E}[B] = \mathbb{E}[\mathbb{E}[\phi(X_1, \ldots, X_k)]; B],$$

or equivalently

$$Y = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] = \mathbb{E}[\phi(X_1, \ldots, X_k)].$$

It suffices to show that Y is independent of $\mathcal{F}_k$. Indeed, by the L^2 characterization of conditional expectation and independence,

$$0 = \mathbb{E}[(\phi(X_1, \ldots, X_k) - Y)Y] = \mathbb{E}[\phi(X_1, \ldots, X_k)]\mathbb{E}[Y] - \mathbb{E}[Y^2] = -\mathrm{Var}[Y],$$

and Y is constant.

1. Since ϕ is bounded, it is integrable and Levy's Downward Thm implies

$$\mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}_n] \to \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}].$$

2. Define

$$A_n(\phi) = \frac{1}{(n)_k} \sum_{1 \le i_1 \ne \cdots \ne i_k \le n} \phi(X_{i_1}, \ldots, X_{i_k}),$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. Note by symmetry

$$A_n(\phi) = \mathbb{E}[A_n(\phi) \,|\, \mathcal{E}_n] = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}_n] \to \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}].$$

3. However, note that

$$\frac{1}{(n)_k} \sum_{1 \in \mathbf{i}} \phi(X_{i_1}, \ldots, X_{i_k}) \leq \frac{k(n-1)_{k-1}}{(n)_k} \sup \phi = \frac{k}{n} \sup \phi \to 0,$$

so that the limit of $A_n(\phi)$ is independent of X_1 and

$$\mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] \in \sigma(X_2, \ldots),$$

and by induction

$$Y = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] \in \sigma(X_{k+1}, \ldots).$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.