PRACTICE MIDTERM 2

1 Answer the following short questions. You do not need to justify your answers.

a) State formally the linear regression problem on input data \( \{(x_i, y_i)\}_{i=1}^n \).

b) Under what conditions on the data is there a unique solution to the linear regression problem?

2 Let \( V = \{v_1, \ldots, v_m\} \) be a list of \( m \) vectors in \( \mathbb{R}^n \). Define
   \[
   W = \left\{ \frac{v_1 - v_2}{\sqrt{2}}, \frac{v_2 - v_3}{\sqrt{2}}, \ldots, \frac{v_{m-1} - v_m}{\sqrt{2}}, v_m \right\}.
   \]
   Assume that \( V \) is an orthonormal list. Prove that the vectors in \( W \) have unit norm, but that \( W \) is not an orthonormal list. Make sure to justify your answer.

3 Let \( f : \mathbb{R} \to \mathbb{R} \) be twice continuously differentiable, let \( a_1, a_2 \) be vectors in \( \mathbb{R}^d \), and let \( b_1, b_2 \in \mathbb{R} \). Consider the following real-valued function of \( x \in \mathbb{R}^d \):
   \[
   g(x) = \frac{1}{2} f(a_1^T x + b_1) + \frac{1}{2} f(a_2^T x + b_2).
   \]
   a) Compute the gradient of \( g \) in vector form in terms of the derivative \( f' \) of \( f \). (By vector form, we mean that it is not enough to write down each element of \( \nabla g(x) \) separately.)

   b) Compute the Hessian of \( g \) in matrix form in terms of the first derivative \( f' \) and second derivative \( f'' \) of \( f \). (By matrix form, we mean that it is not enough to write down each element of \( H_g(x) \) or each of its columns separately.)

4 Let
   \[
   A = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.
   \]
   a) Write \( A \) in outer-product form \( uv^T \).

   b) Use a) to give a compact SVD of \( A \). Make sure to justify your answer.

   c) Give a full SVD of \( A \). Make sure to justify your answer. [Hint: Complete the bases.]

5 Let \( z_1, z_2, z_3 \) be an orthonormal list in \( \mathbb{R}^n \) with \( n > 3 \). Let \( Z \in \mathbb{R}^{n \times 3} \) be the matrix with columns \( z_1, z_2, z_3 \).

   a) Prove that
\[ H = I_{n \times n} - 2ZZ^T, \]

is an orthogonal matrix. Make sure to justify your answer.

b) Given \( \varepsilon \neq 0 \), consider the vector

\[ \mathbf{u} = \frac{1}{\sqrt{1 + \varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \mathbf{z}_1 \end{pmatrix} \in \mathbb{R}^{n+1}. \]

Prove that, if \( \mathbf{O} \in \mathbb{R}^{(n+1) \times (n+1)} \) is an orthogonal matrix, then \( \mathbf{O}\mathbf{u} \) is a unit vector. Make sure to justify your answer. [Hint: First show that \( \mathbf{u} \) is a unit vector.]