*Course:* Math 535 - Mathematical Methods in Data Science (MMiDS) - Spring 2024

# PRACTICE MIDTERM 1

***1*** Give formal definitions of the following terms.

a) Linear subspace of $\mathbb{R}^n$.

b) Rank of a matrix $A \in \mathbb{R}^{n\times m}$.

***2*** Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Compute the orthogonal projection of $\mathbf{v}$ onto $W = \mathrm{span}(\mathbf{w}_1, \mathbf{w}_2)$. Make sure to justify your answer. [*Hint:* Are $\mathbf{w}_1$ and $\mathbf{w}_2$ orthonormal?]

***3*** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Give a formal proof that the following sets are convex.

a) The epigraph of $f$, i.e., the set $\mathbf{epi} f = \{(\mathbf{x}, y) \,:\, \mathbf{x} \in \mathbb{R}^d, y \geq f(\mathbf{x})\}$.

b) The set $G$ of global minimizers of $f$.

***4*** Let $A \in \mathbb{R}^{n\times n}$ be positive definite. Show that the following quantities are strictly positive:

a) all diagonal entries of $A$;

b) the total sum of all entries of $A$.

Make sure to justify your answer.

***5*** Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ be pairwise orthogonal, i.e., for all $i \neq j$ the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal. Assume that, for $i = 1, \ldots, m$, it holds that $\|\mathbf{v}_i\| = i$ (i.e., $\|\mathbf{v}_1\| = 1$, $\|\mathbf{v}_2\| = 2$, $\|\mathbf{v}_3\| = 3$, etc.).

a) Let $V$ be the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Compute $V^T V$. Make sure to justify your answer.

b) Use Gram-Schmidt to compute an orthonormal basis of the subspace $\mathcal{V} = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Make sure to justify your answer.