*Course:* Math 535 - Mathematical Methods in Data Science (MMiDS) - Spring 2024

# PRACTICE FINAL

***1*** Short questions.

a) Consider two matrices $A, B \in \mathbb{R}^{n\times m}$. Suppose that, for $j = 1, \ldots, m$, the $j$-th column of $A$ is a linear combination of the first $j$ columns of $B$. How do we express this as a matrix equation? Choose **one** of the matrix equations below and justify your choice.

(i) $A = GB$ for some upper triangular matrix $G$.

(ii) $A = BH$ for some upper triangular matrix $H$.

(iii) $A = FB$ for some lower triangular matrix $F$.

(iv) $A = BJ$ for some lower triangular matrix $J$.

b) Find a matrix $A$ such that the function

$$f(\mathbf{x}) = \left(x_1, \frac{x_1 + x_2}{2}, x_2, \frac{x_2 + x_3}{2}, x_3, \frac{x_3 + x_4}{2}, x_4, \frac{x_4 + x_5}{2}, x_5\right),$$

can be written as $f(\mathbf{x}) = A\mathbf{x}$ for any vector $\mathbf{x} = (x_1, \ldots, x_5) \in \mathbb{R}^5$.

c) Complete the following sentence: if $\mathbf{v}$ is an eigenvector of $A^T A$ with eigenvalue $\lambda \neq 0$, then **BLANK1** is an eigenvector of $AA^T$ with eigenvalue **BLANK2**.

**2** a) Prove this key property of the spectral decomposition: if $A = Q\Lambda Q^T$ is a spectral decomposition of the symmetric matrix $A \in \mathbb{R}^{n\times n}$, then $A^k = Q\Lambda^k Q^T$ is a spectral decomposition of $A^k$. In particular, show that the eigenvalues of $A^k$ are $\lambda_1^k, \ldots, \lambda_n^k$ if the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$.

b) Use the matrix

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

to show that this last property does not hold for singular values by computing an SVD of $B$ and $B^2$.

c) Suppose the singular values of $C$ are $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Show that the singular values of $CC^TC$ are $\sigma_1^3, \ldots, \sigma_r^3$.

**3** Consider the following $n$ vectors in $\mathbb{R}^n$

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \mathbf{a}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

a) Show that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. What linear subspace are they a basis of?

b) Describe what happens when you run the Gram-Schmidt algorithm to this list of vectors, that is, what $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are produced.

c) Give the matrices $Q$ and $R$ obtained from b).

**4** Let $A \in \mathbb{R}^{n\times m}$ have full column rank. For $B \in \mathbb{R}^{m\times n}$, assume that $I_{n\times n} + AB$ is invertible (i.e., nonsingular).

a) Show that $I_{m\times m} + BA$ is invertible. [*Hint:* Try multiplying $I_{n\times n} + AB$ by $A\mathbf{x}$.]

b) Prove that

$$B(I_{n\times n} + AB)^{-1} = (I_{m\times m} + BA)^{-1}B.$$

[*Hint:* Try multiplying $I_{m\times m} + BA$ by $B$.]

**5** Let $A \in \mathbb{R}^{n \times m}$ have linearly independent columns.

a) Let $X \in \mathbb{R}^{m \times k}$ be a matrix with columns $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^m$ and let $B \in \mathbb{R}^{n \times k}$ be a matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbb{R}^n$. Rewrite

$$||AX - B||_F^2$$

in terms of the columns of $X$ and $B$.

b) Consider the problem of minimizing $||AX - B||_F^2$ over all matrices $X \in \mathbb{R}^{m \times k}$. Show that there is a unique solution $X^*$ and express it in terms of $A$ and $B$ in matrix form.

**6** Let $\mathbf{X} = (X_1, X_2, X_3)$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

a) Compute $f_{X_1, X_2 | X_3}$, i.e., the conditional density of $(X_1, X_2)$ given $X_3$.

b) What is the correlation coefficient between $X_1$ and $X_2$ under the marginal density $f_{X_1, X_2}$ ?