PRACTICE FINAL

1 Short questions.

a) Consider two matrices $A, B \in \mathbb{R}^{n \times m}$. Suppose that, for $j = 1, \ldots, m$, the $j$-th column of $A$ is a linear combination of the first $j$ columns of $B$. How do we express this as a matrix equation? Choose one of the matrix equations below and justify your choice.

(i) $A = GB$ for some upper triangular matrix $G$.

(ii) $A = BH$ for some upper triangular matrix $H$.

(iii) $A = FB$ for some lower triangular matrix $F$.

(iv) $A = BJ$ for some lower triangular matrix $J$.

b) Find a matrix $A$ such that the function

$$f(x) = \left(x_1, \frac{x_1 + x_2}{2}, x_2, \frac{x_2 + x_3}{2}, x_3, \frac{x_3 + x_4}{2}, x_4, \frac{x_4 + x_5}{2}, x_5\right),$$

can be written as $f(x) = Ax$ for any vector $x = (x_1, \ldots, x_5) \in \mathbb{R}^5$.

c) Complete the following sentence: if $v$ is an eigenvector of $A^T A$ with eigenvalue $\lambda \neq 0$, then BLANK1 is an eigenvector of $AA^T$ with eigenvalue BLANK2.
2 a) Prove this key property of the spectral decomposition: if $A = Q \Lambda Q^T$ is a spectral decomposition of the symmetric matrix $A \in \mathbb{R}^{n \times n}$, then $A^k = Q \Lambda^k Q^T$ is a spectral decomposition of $A^k$. In particular, show that the eigenvalues of $A^k$ are $\lambda_1^k, \ldots, \lambda_n^k$ if the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$.

b) Use the matrix 

$$ B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} $$

to show that this last property does not hold for singular values by computing an SVD of $B$ and $B^2$.

c) Suppose the singular values of $C$ are $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Show that the singular values of $CC^T C$ are $\sigma_1^3, \ldots, \sigma_r^3$.

3 Consider the following $n$ vectors in $\mathbb{R}^n$

$$ a_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad a_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. $$

a) Describe what happens when you run the Gram-Schmidt algorithm to this list of vectors, that is, what $q_1, \ldots, q_n$ are produced.

b) Give the matrices $Q$ and $R$ obtained from a).

c) Show that $a_1, \ldots, a_n$ is a basis. What linear subspace is it a basis of?

4 Let $A \in \mathbb{R}^{n \times m}$ have full column rank. For $B \in \mathbb{R}^{m \times n}$, assume that $I_{n \times n} + AB$ is invertible (i.e., nonsingular).

a) Show that $I_{m \times m} + BA$ is invertible. [Hint: Try multiplying $I + AB$ by $Ax$.]

b) Prove that 

$$ B(I + AB)^{-1} = (I + BA)^{-1} B. $$

[Hint: Try multiplying $I + BA$ by $B$.]
5 Let \( A \in \mathbb{R}^{n \times m} \) have linearly independent columns.

a) Let \( X \in \mathbb{R}^{m \times k} \) be a matrix with columns \( x_1, \ldots, x_k \in \mathbb{R}^m \) and let \( B \in \mathbb{R}^{n \times k} \) be a matrix with columns \( b_1, \ldots, b_k \in \mathbb{R}^n \). Rewrite

\[
\|AX - B\|_F^2.
\]

in terms of the columns of \( X \) and \( B \).

b) Consider the problem of minimizing \( \|AX - B\|_F^2 \) over all matrices \( X \in \mathbb{R}^{m \times k} \). Show that there is a unique solution \( X^* \) and express in terms of \( A \) and \( B \) in matrix form.

6 Let \( X = (X_1, X_2, X_3) \) be distributed as \( N_3(\mu, \Sigma) \) where

\[
\mu = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.
\]

a) Compute \( f_{X_1,X_2|X_3} \), i.e., the conditional density of \( (X_1, X_2) \) given \( X_3 \).

b) What is the correlation coefficient between \( X_1 \) and \( X_2 \) under the marginal density \( f_{X_1,X_2} \)?