Modern Discrete Probability

IV - Branching processes

Review

Sébastien Roch
UW–Madison
Mathematics

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Galton-Watson branching processes I

Definition

A *Galton-Watson branching process* is a Markov chain of the following form:

- Let $Z_0 := 1$.
- Let $X(i,t)$, $i \geq 1$, $t \geq 1$, be an array of i.i.d. $\mathbb{Z}_+$-valued random variables with finite mean $m = \mathbb{E}[X(1,1)] < +\infty$, and define inductively,

$$Z_t := \sum_{1 \leq i \leq Z_{t-1}} X(i,t).$$

Galton-Watson branching processes II

Further remarks:

1. The random variable Z_t models the size of a population at time (or generation) t. The random variable $X(i,t)$ corresponds to the number of offspring of the i-th individual (if there is one) in generation $t-1$. Generation t is formed of all offspring of the individuals in generation $t-1$.
2. We denote by $\{p_k\}_{k\geq 0}$ the law of $X(1,1)$. We also let $f(s) := \mathbb{E}[s^{X(1,1)}]$ be the corresponding probability generating function.
3. By tracking genealogical relationships, i.e. who is whose child, we obtain a tree T rooted at the single individual in generation 0 with a vertex for each individual in the progeny and an edge for each parent-child relationship. We refer to T as a *Galton-Watson tree*.

Exponential growth I

Lemma

Let $M_t := m^{-t} Z_t$. Then (M_t) is a nonnegative martingale with respect to the filtration $\mathcal{F}_t = \sigma(Z_0, \ldots, Z_t)$. In particular, $\mathbb{E}[Z_t] = m^t$.

Proof: Recall the following lemma:

Lemma: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $Y_1 = Y_2$ a.s. on $B \in \mathcal{F}$ then $\mathbb{E}[Y_1 \,|\, \mathcal{F}] = \mathbb{E}[Y_2 \,|\, \mathcal{F}]$ a.s. on B.

On $\{Z_{t-1} = k\}$,

$$\mathbb{E}[Z_t \,|\, \mathcal{F}_{t-1}] = \mathbb{E}\left[\sum_{1 \leq j \leq k} X(j,t) \,\middle|\, \mathcal{F}_{t-1} \right] = mk = mZ_{t-1}.$$

This is true for all k. Rearranging shows that (M_t) is a martingale. For the second claim, note that $\mathbb{E}[M_t] = \mathbb{E}[M_0] = 1$. ∎

Exponential growth II

Theorem

We have $M_t \to M_\infty < +\infty$ a.s. for some nonnegative random variable $M_\infty \in \sigma(\cup_t \mathcal{F}_t)$ with $\mathbb{E}[M_\infty] \leq 1$.

Proof: This follows immediately from the martingale convergence theorem for nonnegative martingales and Fatou's lemma. ■

Extinction: some observations I

Observe that 0 is a fixed point of the process. The event

$$\{Z_t \to 0\} = \{\exists t \,:\, Z_t = 0\},$$

is called *extinction*. Establishing when extinction occurs is a central question in branching process theory. We let η be the probability of extinction. *Throughout, we assume that $p_0 > 0$ and $p_1 < 1$.* Here is a first result:

Theorem

A.s. either $Z_t \to 0$ or $Z_t \to +\infty$.

Proof: The process (Z_t) is integer-valued and 0 is the only fixed point of the process under the assumption that $p_1 < 1$. From any state k, the probability of never coming back to $k > 0$ is at least $p_0^k > 0$, so every state $k > 0$ is transient. The claim follows. ∎

Extinction: some observations II

Theorem (Critical branching process)

Assume $m = 1$. Then $Z_t \to 0$ a.s., i.e., $\eta = 1$.

Proof: When $m = 1$, (Z_t) itself is a martingale. Hence (Z_t) must converge to 0 by the corollaries above. ∎

Main result I

Let $f_t(s) = \mathbb{E}[s^{Z_t}]$. Note that, by monotonicity,

$$\eta = \mathbb{P}[\exists t \geq 0 \,:\, Z_t = 0] = \lim_{t \to +\infty} \mathbb{P}[Z_t = 0] = \lim_{t \to +\infty} f_t(0),$$

Moreover, by the Markov property, f_t as a natural recursive form:

$$\begin{aligned} f_t(s) &= \mathbb{E}[s^{Z_t}] \\ &= \mathbb{E}[\mathbb{E}[s^{Z_t} \,|\, \mathcal{F}_{t-1}]] \\ &= \mathbb{E}[f(s)^{Z_{t-1}}] \\ &= f_{t-1}(f(s)) = \cdots = f^{(t)}(s), \end{aligned}$$

where $f^{(t)}$ is the t-th iterate of f.

Main result II

Theorem (Extinction probability)

The probability of extinction η is given by the smallest fixed point of f in $[0,1]$. Moreover:

- (Subcritical regime) *If $m < 1$ then $\eta = 1$.*
- (Supercritical regime) *If $m > 1$ then $\eta < 1$.*

Proof: The case $p_0 + p_1 = 1$ is straightforward: the process dies almost surely after a geometrically distributed time.

So we assume $p_0 + p_1 < 1$ for the rest of the proof.

Main result: proof I

Lemma: On $[0, 1]$, the function f satisfies:

(a) $f(0) = p_0$, $f(1) = 1$;

(b) f is indefinitely differentiable on $[0, 1)$;

(c) f is strictly convex and increasing;

(d) $\lim_{s\uparrow 1} f'(s) = m < +\infty$.

Proof: (a) is clear by definition. The function f is a power series with radius of convergence $R \geq 1$. This implies (b). In particular,

$$f'(s) = \sum_{i\geq 1} i p_i s^{i-1} \geq 0, \quad \text{and} \quad f''(s) = \sum_{i\geq 2} i(i-1) p_i s^{i-2} > 0,$$

because we must have $p_i > 0$ for some $i > 1$ by assumption. This proves (c). Since $m < +\infty$, $f'(1) = m$ is well defined and f' is continuous on $[0, 1]$, which implies (d). ■

Main result: proof II

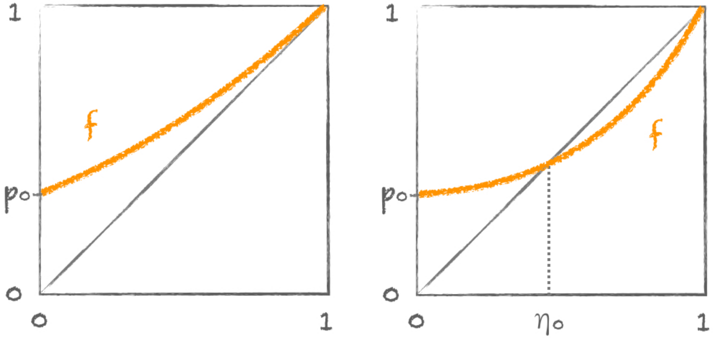
Lemma: We have:

- If $m > 1$ then f has a unique fixed point $\eta_0 \in [0, 1)$.
- If $m < 1$ then $f(t) > t$ for $t \in [0, 1)$. (Let $\eta_0 := 1$ in that case.)

Proof: Assume $m > 1$. Since $f'(1) = m > 1$, there is $\delta > 0$ s.t. $f(1 - \delta) < 1 - \delta$. On the other hand $f(0) = p_0 > 0$ so by continuity of f there must be a fixed point in $(0, 1 - \delta)$. Moreover, by strict convexity and the fact that $f(1) = 1$, if $x \in (0, 1)$ is a fixed point then $f(y) < y$ for $y \in (x, 1)$, proving uniqueness.

The second part follows by strict convexity and monotonicity. ■

Main result: proof III



Main result: proof IV

Lemma: We have:

- If $x \in [0, \eta_0)$, then $f^{(t)}(x) \uparrow \eta_0$
- If $x \in (\eta_0, 1)$ then $f^{(t)}(x) \downarrow \eta_0$

Proof: By monotonicity, for $x \in [0, \eta_0)$, we have $x < f(x) < f(\eta_0) = \eta_0$. Iterating

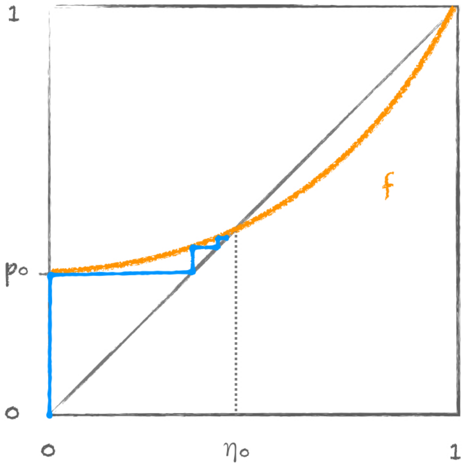
$$x < f^{(1)}(x) < \cdots < f^{(t)}(x) < f^{(t)}(\eta_0) = \eta_0.$$

So $f^{(t)}(x) \uparrow L \leq \eta_0$. By continuity of f we can take the limit inside of

$$f^{(t)}(x) = f(f^{(t-1)}(x)),$$

to get $L = f(L)$. So by definition of η_0 we must have $L = \eta_0$. ■

Main result: proof V



Example: Poisson branching process

Example

Consider the offspring distribution $X(1,1) \sim \mathrm{Poi}(\lambda)$ with $\lambda > 0$. We refer to this case as the *Poisson branching process*. Then

$$f(s) = \mathbb{E}[s^{X(1,1)}] = \sum_{i \geq 0} e^{-\lambda} \frac{\lambda^i}{i!} s^i = e^{\lambda(s-1)}.$$

So the process goes extinct with probability 1 when $\lambda \leq 1$. For $\lambda > 1$, the probability of extinction η_λ is the smallest solution in $[0,1]$ to the equation

$$e^{-\lambda(1-x)} = x.$$

The survival probability $\zeta_\lambda := 1 - \eta_\lambda$ satisfies $1 - e^{-\lambda\zeta_\lambda} = \zeta_\lambda$.

Extinction: back to exponential growth I

Conditioned on extinction, $M_\infty = 0$ a.s.

Theorem

Conditioned on nonextinction, either $M_\infty = 0$ a.s. or $M_\infty > 0$ a.s. In particular, $\mathbb{P}[M_\infty = 0] \in \{\eta, 1\}$.

Proof: A property of rooted trees is said to be *inherited* if all finite trees satisfy this property and whenever a tree satisfies the property then so do all the descendant trees of the children of the root. The property $\{M_\infty = 0\}$ is inherited. The result then follows from the following 0-1 law.

Lemma: For a Galton-Watson tree T, an inherited property A has, conditioned on nonextinction, probability 0 or 1.
Proof of lemma: Let $T^{(1)}, \ldots, T^{(Z_1)}$ be the descendant subtrees of the children of the root. Then, by independence,

$$\mathbb{P}[A] = \mathbb{E}[\mathbb{P}[T \in A \,|\, Z_1]] \leq \mathbb{E}[\mathbb{P}[T^{(i)} \in A, \forall i \leq Z_1 \,|\, Z_1]] = \mathbb{E}[\mathbb{P}[A]^{Z_1}] = f(\mathbb{P}[A]),$$

so $\mathbb{P}[A] \in [0, \eta] \cup \{1\}$. Also $\mathbb{P}[A] \geq \eta$ because A holds for finite trees. ■■

Extinction: back to exponential growth II

Theorem

Let (Z_t) be a branching process with $m = \mathbb{E}[X(1,1)] > 1$ and $\sigma^2 = \mathrm{Var}[X(1,1)] < +\infty$. Then, (M_t) converges in L^2 and, in particular, $\mathbb{E}[M_\infty] = 1$.

Proof: From the orthogonality of increments

$$\mathbb{E}[M_t^2] = \mathbb{E}[M_{t-1}^2] + \mathbb{E}[(M_t - M_{t-1})^2].$$

On $\{Z_{t-1} = k\}$

$$\begin{aligned}
\mathbb{E}[(M_t - M_{t-1})^2 \,|\, \mathcal{F}_{t-1}] &= m^{-2t}\mathbb{E}[(Z_t - mZ_{t-1})^2 \,|\, \mathcal{F}_{t-1}] \\
&= m^{-2t}\mathbb{E}\left[\left(\sum_{i=1}^{k} X(i,t) - mk\right)^2 \,\middle|\, \mathcal{F}_{t-1}\right] \\
&= m^{-2t}k\sigma^2 \\
&= m^{-2t}Z_{t-1}\sigma^2.
\end{aligned}$$

Extinction: back to exponential growth III

Hence

$$\mathbb{E}[M_t^2] = \mathbb{E}[M_{t-1}^2] + m^{-t-1}\sigma^2.$$

Since $\mathbb{E}[M_0^2] = 1$,

$$\mathbb{E}[M_t^2] = 1 + \sigma^2 \sum_{i=2}^{t+1} m^{-i},$$

which is uniformly bounded when $m > 1$. So (M_t) converges in L^2. Finally by Fatou's lemma

$$\mathbb{E}|M_\infty| \leq \sup \|M_t\|_1 \leq \sup \|M_t\|_2 < +\infty$$

and

$$|\mathbb{E}[M_t] - \mathbb{E}[M_\infty]| \leq \|M_t - M_\infty\|_1 \leq \|M_t - M_\infty\|_2,$$

implies the convergence of expectations. ■

Exploration process I

We consider an exploration process of the Galton-Watson tree T. The exploration process, started at the root 0, has 3 types of vertices:

- $\mathcal{A}_t$: *active*, $\mathcal{E}_t$: *explored*, $\mathcal{N}_t$: *neutral*.

We start with $\mathcal{A}_0 := \{0\}$, $\mathcal{E}_0 := \emptyset$, and $\mathcal{N}_0$ contains all other vertices in T. At time t, if $\mathcal{A}_{t-1} = \emptyset$ we let $(\mathcal{A}_t, \mathcal{E}_t, \mathcal{N}_t) := (\mathcal{A}_{t-1}, \mathcal{E}_{t-1}, \mathcal{N}_{t-1})$. Otherwise, we pick an element, a_t, from $\mathcal{A}_{t-1}$ and set:

- $\mathcal{A}_t := \mathcal{A}_{t-1} \cup \{x \in \mathcal{N}_{t-1} \,:\, \{x, a_t\} \in T\} \backslash \{a_t\}$,
- $\mathcal{E}_t := \mathcal{E}_{t-1} \cup \{a_t\}$,
- $\mathcal{N}_t := \mathcal{N}_{t-1} \backslash \{x \in \mathcal{N}_{t-1} \,:\, \{x, a_t\} \in T\}$.

To be concrete, we choose a_t in breadth-first search (or first-come-first-serve) manner: we exhaust all vertices in generation t before considering vertices in generation $t+1$.

Exploration process II

We imagine revealing the edges of T as they are encountered in the exploration process and we let $(\mathcal{F}_t)$ be the corresponding filtration. In words, starting with 0, the Galton-Watson tree T is progressively grown by adding to it at each time a child of one of the previously explored vertices and uncovering its children in T. In this process, $\mathcal{E}_t$ is the set of previously explored vertices and $\mathcal{A}_t$ is the set of vertices who are known to belong to T but whose full neighborhood is waiting to be uncovered. The rest of the vertices form the set $\mathcal{N}_t$.

Exploration process III

Let $A_t := |\mathcal{A}_t|$, $E_t := |\mathcal{E}_t|$, and $N_t := |\mathcal{N}_t|$. Note that (E_t) is non-decreasing while (N_t) is non-increasing. Let

$$\tau_0 := \inf\{t \geq 0 \,:\, A_t = 0\},$$

(which by convention is $+\infty$ if there is no such t). The process is fixed for all $t > \tau_0$. Notice that $E_t = t$ for all $t \leq \tau_0$, as exactly one vertex is explored at each time until the set of active vertices is empty.

Lemma

Let W be the total progeny. Then

$$W = \tau_0.$$

Random walk representation I

The process (A_t) admits a simple recursive form. Recall that $A_0 := 1$. Conditioning on $\mathcal{F}_{t-1}$:

- If $A_{t-1} = 0$, the exploration process has finished its course and $A_t = 0$. Otherwise, (a) one active vertex becomes an explored vertex and (b) its neutral neighbors become active vertices. That is,

$$A_t = \begin{cases} A_{t-1} + \Big[\underbrace{-1}_{\text{(a)}} + \underbrace{X_t}_{\text{(b)}}\Big], & t-1 < \tau_0, \\ 0, & \text{o.w.} \end{cases}$$

  where X_t is distributed according to the offspring distribution.

Random walk representation II

We let $Y_t = X_t - 1 \geq -1$ and

$$S_t := 1 + \sum_{i=1}^{t} Y_i,$$

with $S_0 := 1$. Then

$$\begin{aligned}\tau_0 &= \inf\{t \geq 0 \;:\; S_t = 0\} \\ &= \inf\{t \geq 0 \;:\; 1 + [X_1 - 1] + \cdots + [X_t - 1] = 0\} \\ &= \inf\{t \geq 0 \;:\; X_1 + \cdots + X_t = t - 1\},\end{aligned}$$

and (A_t) is a random walk started at 1 with steps (Y_t) stopped when it hits 0 for the first time:

$$A_t = (S_{t \wedge \tau_0}).$$

Duality principle I

Theorem

Let (Z_t) be a branching process with offspring distribution $\{p_k\}_{k\geq 0}$ and extinction probability $\eta < 1$. Let (Z'_t) be a branching process with offspring distribution $\{p'_k\}_{k\geq 0}$ where

$$p'_k = \eta^{k-1} p_k.$$

Then (Z_t) conditioned on extinction has the same distribution as (Z'_t), which is referred to as the dual branching process.

Duality principle II

Some remarks:

- Note that

$$\sum_{k\geq 0} p'_k = \sum_{k\geq 0} \eta^{k-1} p_k = \eta^{-1} f(\eta) = 1,$$

because η is a fixed point of f. So $\{p'_k\}_{k\geq 0}$ is indeed a probability distribution.

- Note further that

$$\sum_{k\geq 0} k p'_k = \sum_{k\geq 0} k \eta^{k-1} p_k = f'(\eta) < 1,$$

since f' is strictly increasing, $f(\eta) = \eta < 1$ and $f(1) = 1$. So the dual branching process is subcritical.

Duality principle III

Proof: We use the random walk representation. Let $H = (X_1, \ldots, X_{\tau_0})$ and $H' = (X'_1, \ldots, X'_{\tau'_0})$ be the *histories* of the processes (Z_t) and (Z'_t) respectively. (Under breadth-first search, the process (Z_t) can be reconstructed from H.) In the case of extinction, the history of (Z_t) has finite length. We call $(x_1, \ldots, x_t)$ a *valid history* if $x_1 + \cdots + x_i - (i-1) > 0$ for all $i < t$ and $x_1 + \cdots + x_t - (t-1) = 0$. By definition of the conditional probability, for a valid history $(x_1, \ldots, x_t)$ with a finite t,

$$\mathbb{P}[H = (x_1, \ldots, x_t) \,|\, \tau_0 < +\infty] = \frac{\mathbb{P}[H = (x_1, \ldots, x_t)]}{\mathbb{P}[\tau_0 < +\infty]} = \eta^{-1} \prod_{i=1}^{t} p_{x_i}.$$

Because $x_1 + \cdots + x_t = t - 1$,

$$\eta^{-1} \prod_{i=1}^{t} p_{x_i} = \eta^{-1} \prod_{i=1}^{t} \eta^{1-x_i} p'_{x_i} = \prod_{i=1}^{t} p'_{x_i} = \mathbb{P}[H' = (x_1, \ldots, x_t)].$$

■

Duality principle: example

Example (Poisson branching process)

Let (Z_t) be a Galton-Watson branching process with offspring distribution $\mathrm{Poi}(\lambda)$ where $\lambda > 1$. Then the dual probability distribution is given by

$$p'_k = \eta^{k-1} p_k = \eta^{k-1} e^{-\lambda} \frac{\lambda^k}{k!} = \eta^{-1} e^{-\lambda} \frac{(\lambda\eta)^k}{k!},$$

where recall that $e^{-\lambda(1-\eta)} = \eta$, so

$$p'_k = e^{\lambda(1-\eta)} e^{-\lambda} \frac{(\lambda\eta)^k}{k!} = e^{-\lambda\eta} \frac{(\lambda\eta)^k}{k!}.$$

That is, the dual branching process has offspring distribution $\mathrm{Poi}(\lambda\eta)$.

Hitting-time theorem

Theorem

Let (Z_t) be a Galton-Watson branching process with total progeny W. In the random walk representation of (Z_t),

$$\mathbb{P}[W = t] = \frac{1}{t}\mathbb{P}[X_1 + \cdots + X_t = t - 1],$$

for all $t \geq 1$.

Note that this formula is rather remarkable as the probability on the l.h.s. is $\mathbb{P}[S_i > 0, \forall i < t \text{ and } S_t = 0]$ while the probability on the r.h.s. is $\mathbb{P}[S_t = 0]$.

Spitzer's combinatorial lemma I

We start with a lemma of independent interest. Let $u_1, \ldots, u_t \in \mathbb{R}$ and define $r_0 := 0$ and $r_i := u_1 + \cdots + u_i$ for $1 \leq i \leq t$. We say that j is a *ladder index* if $r_j > r_0 \vee \cdots \vee r_{j-1}$. Consider the cyclic permutations of $\boldsymbol{u} = (u_1, \ldots, u_t)$: $\boldsymbol{u}^{(0)} = \boldsymbol{u}$, $\boldsymbol{u}^{(1)} = (u_2, \ldots, u_t, u_1)$, ..., $\boldsymbol{u}^{(t-1)} = (u_t, u_1, \ldots, u_{t-1})$. Define the corresponding partial sums $r_j^{(\beta)} := u_1^{(\beta)} + \cdots + u_j^{(\beta)}$ for $j = 1, \ldots, t$ and $\beta = 0, \ldots, t-1$. Observe that

$$
\begin{aligned}
(r_1^{(\beta)}, \ldots, r_t^{(\beta)}) \\
&= (r_{\beta+1} - r_\beta, r_{\beta+2} - r_\beta, \ldots, r_t - r_\beta, \\
&\qquad [r_t - r_\beta] + r_1, [r_t - r_\beta] + r_2, \ldots, [r_t - r_\beta] + r_\beta) \\
&= (r_{\beta+1} - r_\beta, r_{\beta+2} - r_\beta, \ldots, r_t - r_\beta, \\
&\qquad r_t - [r_\beta - r_1], r_t - [r_\beta - r_2], \ldots, r_t - [r_\beta - r_{\beta-1}], r_t) \quad (1)
\end{aligned}
$$

Spitzer's combinatorial lemma II

Lemma

Assume $r_t > 0$. Let ℓ be the number of cyclic permutations such that t is a ladder index. Then $\ell \geq 1$. Moreover, each such cyclic permutation has exactly ℓ ladder indices.

Proof: We first show that $\ell \geq 1$, i.e., there is at least one cyclic permutation where t is a ladder index. Let β be the smallest index achieving the maximum of $r_1, \ldots, r_t$, i.e.,

$$r_\beta > r_1 \vee \cdots \vee r_{\beta-1} \quad \text{and} \quad r_\beta \geq r_{\beta+1} \vee \cdots \vee r_t.$$

From (1),

$$r_{\beta+i} - r_\beta \leq 0 < r_t, \qquad \forall i = 1, \ldots, t-\beta,$$

and

$$r_t - [r_\beta - r_j] < r_t, \qquad \forall j = 1, \ldots, \beta - 1.$$

Moreover, $r_t > 0 = r_0$ by assumption. So, in $\boldsymbol{u}^{(\beta)}$, t is a ladder index.

Spitzer's combinatorial lemma III

Since $\ell \geq 1$, we can assume w.l.o.g. that $\boldsymbol{u}$ is such that t is a ladder index. Then β is a ladder index in $\boldsymbol{u}$ if and only if

$$r_\beta > r_0 \vee \cdots \vee r_{\beta-1},$$

if and only if

$$r_t > r_t - r_\beta \qquad \text{and} \qquad r_t - [r_\beta - r_j] < r_t,\ \forall j = 1, \ldots, \beta - 1.$$

Moreover, because $r_t > r_j$ for all j, we have $r_t - [r_{\beta+i} - r_\beta] = (r_t - r_{\beta+i}) + r_\beta$ and the last equation is equivalent to

$$r_t > r_t - [r_{\beta+i} - r_\beta],\ \forall i = 1, \ldots, t - \beta \quad \text{and} \quad r_t - [r_\beta - r_j] < r_t,\ \forall j = 1, \ldots, \beta - 1.$$

That is, t is a ladder index in the β-th cyclic permutation. ■

Back to the hitting-time theorem: proof I

Proof: Let $R_i := 1 - S_i$ and $U_i := 1 - X_i$ for all $i = 1, \ldots, t$ and let $R_0 := 0$. Then

$$\{X_1 + \cdots + X_t = t - 1\} = \{R_t = 1\},$$

and

$$\{W = t\} = \{t \text{ is the first ladder index in } R_1, \ldots, R_t\}.$$

By symmetry, for all β

$$\begin{aligned} &\mathbb{P}[t \text{ is the first ladder index in } R_1, \ldots, R_t] \\ &\quad = \mathbb{P}[t \text{ is the first ladder index in } R_1^{(\beta)}, \ldots, R_t^{(\beta)}]. \end{aligned}$$

Let $\mathcal{E}_\beta$ be the event on the last line. Hence

$$\mathbb{P}[W = t] = \mathbb{E}[\mathbb{1}_{\mathcal{E}_1}] = \frac{1}{t}\mathbb{E}\left[\sum_{\beta=1}^{t} \mathbb{1}_{\mathcal{E}_\beta}\right]$$

Back to the hitting-time theorem: proof II

Proof: By Spitzer's combinatorial lemma, there is at most one cyclic permutation where t is the first ladder index. In particular, $\sum_{\beta=1}^{t} \mathbb{1}_{\mathcal{E}_\beta} \in \{0,1\}$. So

$$\mathbb{P}[W = t] = \frac{1}{t}\mathbb{P}\left[\cup_{\beta=1}^{t}\mathcal{E}_\beta\right].$$

Finally observe that, because $R_0 = 0$ and $U_i \leq 1$ for all i, the partial sum at the j-th ladder index must take value j. So the event $\{\cup_{\beta=1}^{t}\mathcal{E}_\beta\}$ implies that $\{R_t = 1\}$ because the last partial sum of all cyclic permutations is R_t. Similarly, because there is at least one cyclic permutation such that t is a ladder index, the event $\{R_t = 1\}$ implies $\{\cup_{\beta=1}^{t}\mathcal{E}_\beta\}$. Therefore,

$$\mathbb{P}[W = t] = \frac{1}{t}\mathbb{P}\left[R_t = 1\right],$$

which concludes the proof.

■

Hitting-time theorem: example

Example (Poisson branching process)

Let (Z_t) be a Galton-Watson branching process with offspring distribution $\mathrm{Poi}(\lambda)$ where $\lambda > 0$. Let W be its total progeny. By the hitting-time theorem, for $t \geq 1$,

$$\begin{aligned} \mathbb{P}[W = t] &= \frac{1}{t}\mathbb{P}[X_1 + \cdots + X_t = t - 1] \\ &= \frac{1}{t} e^{-\lambda t} \frac{(\lambda t)^{t-1}}{(t-1)!} \\ &= e^{-\lambda t} \frac{(\lambda t)^{t-1}}{t!}, \end{aligned}$$

where we used that a sum of independent Poisson is Poisson.

Bond percolation on Galton-Watson trees I

Let T be a Galton-Watson tree for an offspring distribution with mean $m > 1$. Perform bond percolation on T with density p.

Theorem

Conditioned on nonextinction,

$$p_c(T) = \frac{1}{m} \qquad a.s.$$

Proof: Let $\mathcal{C}_0$ be the cluster of the root in T with density p. We can think of $\mathcal{C}_0$ as being generated by a Galton-Watson branching process where the offspring distribution is the law of $\sum_{i=1}^{X(1,1)} I_i$ where the I_is are i.i.d. $\mathrm{Ber}(p)$ and $X(1,1)$ is distributed according to the offspring distribution of T. In particular, by conditioning on $X(1,1)$, the offspring mean under $\mathcal{C}_0$ is mp. If $mp \leq 1$ then

$$1 = \mathbb{P}_p[|\mathcal{C}_0| < +\infty] = \mathbb{E}[\mathbb{P}_p[|\mathcal{C}_0| < +\infty \mid T]],$$

and we must have $\mathbb{P}_p[|\mathcal{C}_0| < +\infty \mid T] = 1$ a.s. In other words, $p_c(T) \geq \frac{1}{m}$ a.s.

Bond percolation on Galton-Watson trees II

On the other hand, the property of trees $\{\mathbb{P}_p[|\mathcal{C}_0| < +\infty \,|\, T] = 1\}$ is inherited. So by our previous lemma, conditioned on nonextinction, it has probability 0 or 1. That probability is of course 1 on extinction. So by

$$\mathbb{P}_p[|\mathcal{C}_0| < +\infty] = \mathbb{E}[\mathbb{P}_p[|\mathcal{C}_0| < +\infty \,|\, T]],$$

if the probability is 1 conditioned on nonextinction then it must be that $mp \leq 1$. In other words, for any fixed p such that $mp > 1$, conditioned on nonextinction $\mathbb{P}_p[|\mathcal{C}_0| < +\infty \,|\, T] = 0$ a.s. By monotonicity of $\mathbb{P}_p[|\mathcal{C}_0| < +\infty \,|\, T]$ in p, taking a limit $p_n \to 1/m$ proves the result.