1 Overview

In the last lecture we completed the proof of the Cramer-Rao bound and discussed a few examples. In this lecture we will begin our foray into High Dimensional Probability by Roman Vershynin [1] and discuss sub-gaussian random variables and some of their properties.

2 Sub-Gaussian Random Variables

These are a very useful class of random variables whose tails decay at least as fast as Gaussian random variables.

Notation:

- $a \lesssim b : \exists C > 0 \text{ s.t } a \leq Cb$
- $a \vee b := \max\{a, b\}$

Proposition 1. (2.12 in [1]) Let $g \sim N(0, 1)$ i.e., $f_g(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$. Then, for any $t \geq 0$,

$$\left(1 - \frac{1}{t^3}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq P(g \geq t) \leq \left(1 - \frac{1}{t}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad (1)$$

Definition 2. The **sub-gaussian norm** of $X \in \mathbb{R}$ is

$$\|X\|_{\psi_2} := \inf\left\{ t \geq 0 : \mathbb{E} \left[ \psi_2\left( \frac{|X|}{t} \right) \right] \leq 1 \right\} \quad (2)$$

where $\psi_2(x) = e^{x^2} - 1$

Remarks:

- The sub-gaussian norm is a valid norm and therefore obeys useful properties such as absolute homogeneity and the triangle inequality.
- The “-1” is a convention chosen so that $\psi_2(0) = 0$
- Refer [1] for a discussion on the connection with $\ell_p$ spaces.
2.1 Sub-gaussian norms of some examples

**Gaussian case:** \( g \sim N(0, 1) \).

First note that we can push the “−1” to the other side of the inequality in Equation (2) and therefore, it suffices to consider

\[
E \left[ \psi_2 \left( \frac{|g|}{t} \right) \right] = E \left[ \exp \left( \frac{g^2}{t^2} \right) \right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} + \frac{x^2}{t^2} \right) dx
\]

It is clear that the above integral is finite if and only if \( t \geq \sqrt{2} \). Moreover, \( E[\psi_2(|g|/t)] \to 1 \) as \( t \to \infty \) as the integral reduces to the integral of the Gaussian pdf. Therefore, \( \exists \ t^* > \sqrt{2} \) such that \( E[\psi_2(|g|/t)] \leq 2 \) so that \( \|g\|_{\psi_2} = t^* \).

**Bounded case:** \( X \in [a, b] \) a.s.

Then, \( \|X\|_{\psi_2} \lesssim |a| \lor |b| \)

2.2 Useful properties of the sub-gaussian norm

**Exercise (2.5.7 in [1])** Show that

\[
\|X + Y\|_{\psi_2} \leq \|X\|_{\psi_2} + \|Y\|_{\psi_2}
\]

**Proof.** Note that,

\[
\psi_2 \left( \frac{|X + Y|}{a + b} \right) \leq \psi_2 \left( \frac{|X| + |Y|}{a + b} \right) \leq \frac{a}{a + b} \psi_2 \left( \frac{|X|}{b} \right) + \frac{b}{a + b} \psi_2 \left( \frac{|Y|}{b} \right)
\]

where the first inequality holds since \( \psi_2 \) is an increasing function of its argument and the second is from applying Jensen’s inequality since \( \psi_2 \) is convex.

Therefore, for any \( a > \|X\|_{\psi_2}, b > \|Y\|_{\psi_2} \)

\[
E \left[ \left( \frac{|X + Y|}{a + b} \right) \right] \leq \frac{a}{a + b} E \left[ \psi_2 \left( \frac{|X|}{a} \right) \right] + \frac{b}{a + b} E \left[ \psi_2 \left( \frac{|Y|}{b} \right) \right] \leq 1
\]

where both expectations are bounded by 1 simply by applying the definition of sub-gaussian norm.

**Theorem 3. (2.5.2 in [1]).** If \( \|X\|_{\psi_2} < +\infty \), then the following properties are equivalent

(i) \( \mathbb{P}(\{|X| > t\}) \leq 2 \exp \left( -\frac{ct^2}{K^2} \right), \text{ for all } t \geq 0. \)

(ii) \( \|X\|_p = (\mathbb{E}(|X|^p))^{\frac{1}{p}} \leq C \cdot K \sqrt{p}, \text{ for all } p \geq 1. \)
(v) If $\mathbb{E}X = 0$, then $\mathbb{E}[\exp(\lambda X)] \leq \exp(C\lambda^2 K^2)$, for all $\lambda \in \mathbb{R}$

where $K = \|X\|_{\psi_2}$, $c$ is a small universal constant and $C$ is a large universal constant. Vice Versa, if any of the above properties is true, then $\|X\|_{\psi_2} \lesssim K$.

Proof ideas.
(i) $\Rightarrow$ (ii): $\mathbb{E}|X|^p = \int_0^\infty \mathbb{P}(|X| \geq t)t^{p-1}dt = \ldots$ The $\sqrt{p}$ should come naturally out of the calculus.

(ii) $\Rightarrow \|X\|_{\psi_2} < +\infty$: Using the Taylor’s expansion, $\mathbb{E}[\exp(\lambda X^2)] = 1 + \sum_{p=1}^\infty \frac{\lambda^{2p}\mathbb{E}[X^{2p}]}{p!} = \ldots$. Applying property (ii) and massaging the terms should give the result.

$\|X\|_{\psi_2} < +\infty \Rightarrow$ (i): For simplicity, assume that $\|X\|_{\psi_2} = 1$. The proof is identical otherwise with just appropriate scaling.

$$
\mathbb{P}(|X| \geq t) = \mathbb{P}(e^{X^2} \geq e^{t^2}) \\
\leq e^{-t^2} \mathbb{E}[e^{X^2}] \quad \text{(Markov’s inequality)} \\
\leq 2e^{-t^2}
$$

(v) $\Rightarrow$ (i):

$$
\mathbb{P}(X \geq t) = \mathbb{P}(e^{\lambda x} \geq e^{\lambda t}), \lambda > 0 \\
\leq e^{-\lambda t} \mathbb{E}[e^{\lambda X}] \quad \text{(Markov’s inequality)} \\
\leq e^{-\lambda t + C\lambda^2} \quad \text{(applying (v))} \\
= e^{-t^2/4c}
$$

where the last equality follows by choosing a good $\lambda$.

References