Notes 27 : Brownian motion: path properties

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References:[Dur10, Section 8.1], [MP10, Section 1.1, 1.2, 1.3].

Recall:

DEF 27.1 (Covariance) *Let $X = (X_1, \ldots, X_d)$ be a random vector with mean $\mu = \mathbb{E}[X]$. The* covariance of X *is the $d \times d$ matrix Γ with entries*

$$\Gamma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)].$$

DEF 27.2 (Gaussian distribution) *A* standard Gaussian *is a RV Z with CF*

$$\phi_Z(t) = \exp\left(-t^2/2\right),$$

and density

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right).$$

In particular, Z has mean 0 and variance 1. More generally,

$$X = \sigma Z + \mu,$$

is a Gaussian RV with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

DEF 27.3 (Multivariate Gaussian) *A d-dimensional standard Gaussian is a random vector $X = (X_1, \ldots, X_d)$ where the X_is are independent standard Gaussians. In particular, X has mean 0 and covariance matrix I. More generally, a random vector $X = (X_1, \ldots, X_d)$ is Gaussian if there is a vector b, a $d \times r$ matrix A and an r-dimensional standard Gaussian Y such that*

$$X = AY + b.$$

Then X has mean $\mu = b$ and covariance matrix $\Gamma = AA^T$. The CF of X is given by

$$\phi_X(t) = \exp\left(i \sum_{j=1}^{d} t_j \mu_j - \frac{1}{2} \sum_{j,k=1}^{d} t_j t_k \Gamma_{jk} \right).$$

COR 27.4 (Independence) *Let $X = (X_1, \ldots, X_d)$ be a multivariate Gaussian. Then the X_is are independent if and only if $\Gamma_{ij} = 0$ for all $i \neq j$, that is, if they are* uncorrelated.

COR 27.5 (Linear combinations) *The random vector $(X_1, \ldots, X_d)$ is multivariate Gaussian if and only if all linear combinations of its components are Gaussian.*

DEF 27.6 (Gaussian process) *A continuous-time stochastic process $\{X(t)\}_{t\geq 0}$ is a* Gaussian process *if for all $n \geq 1$ and $0 \leq t_1 < \cdots < t_n < +\infty$ the random vector*

$$(X(t_1), \ldots, X(t_n)),$$

is multivariate Gaussian. The mean and covariance functions of X are $\mathbb{E}[X(t)]$ and $\mathrm{Cov}[X(s), X(t)]$ respectively.

DEF 27.7 (Brownian motion: Definition I) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X is a Gaussian process with almost surely continuous paths, that is,*

$$\mathbb{P}[X(t) \textit{ is continuous in } t] = 1,$$

such that $X(0) = 0$,

$$\mathbb{E}[X(t)] = 0,$$

and

$$\mathrm{Cov}[X(s), X(t)] = s \wedge t.$$

More generally, $B = \sigma X + x$ is a Brownian motion started at x.

DEF 27.8 (Stationary independent increments) *An SP $\{X(t)\}_{t\geq 0}$ has* stationary increments *if the distribution of $X(t) - X(s)$ depends only on $t - s$ for all $0 \leq s \leq t$. It has* independent increments *if the RVs $\{X(t_{j+1} - X(t_j)), 1 \leq j < n\}$ are independent whenever $0 \leq t_1 < t_2 < \cdots < t_n$ and $n \geq 1$.*

DEF 27.9 (Brownian motion: Definition II) *The continuous-time stochastic process $X = \{X(t)\}_{t\geq 0}$ is a* standard Brownian motion *if X has almost surely continuous paths and stationary independent increments such that $X(s+t) - X(s)$ is Gaussian with mean 0 and variance t.*

THM 27.10 (Existence) *Standard Brownian motion $B = \{B(t)\}_{t\geq 0}$ exists.*

1 Invariance

We begin with some useful invariance properties. The following are immediate.

THM 27.11 (Time translation) *Let $s \geq 0$. If $B(t)$ is a standard Brownian motion, then so is $X(t) = B(t+s) - B(s)$.*

THM 27.12 (Scaling invariance) *Let $a > 0$. If $B(t)$ is a standard Brownian motion, then so is $X(t) = a^{-1}B(a^2 t)$.*

Proof: *(Sketch)* We compute the variance of the increments:

$$\begin{aligned}\mathrm{Var}[X(t) - X(s)] &= \mathrm{Var}[a^{-1}(B(a^2 t) - B(a^2 s))] \\ &= a^{-2}(a^2 t - a^2 s) \\ &= t - s.\end{aligned}$$

■

THM 27.13 (Time inversion) *If $B(t)$ is a standard Brownian motion, then so is*

$$X(t) = \begin{cases} 0, & t = 0, \\ tB(t^{-1}), & t > 0. \end{cases}$$

Proof: *(Sketch)* We compute the covariance function for $s < t$:

$$\begin{aligned}\mathrm{Cov}[X(s), X(t)] &= \mathrm{Cov}[sB(s^{-1}), tB(t^{-1})] \\ &= st\left(s^{-1} \wedge t^{-1}\right) \\ &= s.\end{aligned}$$

It remains to check continuity at 0. Note that

$$\left\{\lim_{t \downarrow 0} B(t) = 0\right\} = \bigcap_{m \geq 1} \bigcup_{n \geq 1} \left\{|B(t)| \leq 1/m,\ \forall t \in \mathbb{Q} \cap (0, 1/n)\right\},$$

and

$$\left\{\lim_{t \downarrow 0} X(t) = 0\right\} = \bigcap_{m \geq 1} \bigcup_{n \geq 1} \left\{|X(t)| \leq 1/m,\ \forall t \in \mathbb{Q} \cap (0, 1/n)\right\}.$$

(We are using continuity on $t > 0$.) The RHSs have the same probability because the distributions on all finite-dimensional sets (including 0)—and therefore on the rationals—are the same. The LHS of the first one has probability 1. ■

Typical applications of these are:

COR 27.14 *For $a < 0 < b$, let*

$$T(a, b) = \inf\{t \geq 0 \,:\, B(t) \in \{a, b\}\}.$$

Then

$$\mathbb{E}[T(a, b)] = a^2\mathbb{E}[T(1, b/a)].$$

In particular, $\mathbb{E}[T(-b, b)]$ is a constant multiple of b^2.

Proof: Let $X(t) = a^{-1}B(a^2 t)$. Then,

$$\begin{aligned}\mathbb{E}[T(a, b)] &= a^2\mathbb{E}[\inf\{t \geq 0, : X(t) \in \{1, b/a\}\}]\\ &= a^2\mathbb{E}[T(1, b/a)].\end{aligned}$$

■

COR 27.15 *Almost surely,*

$$t^{-1}B(t) \to 0.$$

Proof: Let $X(t)$ be the time inversion of $B(t)$. Then

$$\lim_{t\to\infty} \frac{B(t)}{t} = \lim_{t\to\infty} X(1/t) = X(0) = 0.$$

■

2 Modulus of continuity

By construction, $B(t)$ is continuous a.s. In fact, we can prove more.

DEF 27.16 (Hölder continuity) *A function f is said* locally α-Hölder continuous at x *if there exists $\varepsilon > 0$ and $c > 0$ such that*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for all y with $|y - x| < \varepsilon$. We refer to α as the Hölder exponent and to c as the Hölder constant.

THM 27.17 (Holder continuity) *If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally α-Hölder continuous.*

Proof:

LEM 27.18 *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \leq t \leq 1-h$,*

$$|B(t+h) - B(t)| \leq C\sqrt{h \log(1/h)}.$$

Proof: Recall our construction of Brownian motion on $[0,1]$. Let

$$\mathcal{D}_n = \{k2^{-n} \,:\, 0 \leq k \leq 2^n\},$$

and

$$\mathcal{D} = \cup_{n=0}^{\infty} \mathcal{D}_n.$$

Note that $\mathcal{D}$ is countable and consider $\{Z_t\}_{t\in\mathcal{D}}$ a collection of independent standard Gaussians. Let

$$F_0(t) = \begin{cases} Z_1, & t = 1, \\ 0, & t = 0, \\ \text{linearly}, & \text{in between.} \end{cases}$$

and for $n \geq 1$

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t, & t \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}, \\ 0, & t \in \mathcal{D}_{n-1}, \\ \text{linearly}, & \text{in between.} \end{cases}$$

Finally

$$B(t) = \sum_{n=0}^{\infty} F_n(t).$$

Each F_n is piecewise linear and its derivative exists almost everywhere. By construction, we have

$$\|F_n'\|_\infty \leq \frac{\|F_n\|_\infty}{2^{-n}}.$$

Recall that there is N (random) such that $|Z_d| < c\sqrt{n}$ for all $d \in \mathcal{D}_n$ with $n > N$. In particular, for $n > N$ we have

$$\|F_n\|_\infty < c\sqrt{n} 2^{-(n+1)/2}.$$

Using the mean-value theorem, assuming $l > N$,

$$\begin{aligned} |B(t+h) - B(t)| &\leq \sum_{n=0}^{\infty} |F_n(t+h) - F_n(t)| \\ &\leq \sum_{n=0}^{l} h\|F_n'\|_\infty + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty, \\ &\leq h\sum_{n=0}^{N} \|F_n'\|_\infty + ch\sum_{n=N}^{l} \sqrt{n}2^{n/2} + 2c\sum_{n=l+1}^{\infty} \sqrt{n}2^{-n/2}. \end{aligned}$$

(The idea above is that the sup norm and the sup norm of the derivatives by themselves are not good enough. But each is good in its own domain: derivative for small n because of the h, sup norm for large n because the series is summable. You need to combina them and find the right breakpoint, that is, when both are essentially equal.) Take h small enough that the first term is smaller than $\sqrt{h \log(1/h)}$ and l defined by $2^{-l} < h \leq 2^{-l+1}$ exceeds N. Then approximating the second and third terms by their largest element gives the result. ■

We go back to the proof of the theorem. For each k, we can find an $h(k)$ small enough so that the result applies to the standard BMs

$$\{B(k+t) - B(k) \,:\, t \in [0,1]\},$$

and

$$\{B(k+1-t) - B(k+1) \,:\, t \in [0,1]\}.$$

(By the same kind of invariance arguments we used before, time reversal preserves standard BM. We need the time reversal because the theorem is stated only for increments in one direction.) Since there are countably many intervals $[k, k+1)$, such $h(k)$'s exist almost surely on all intervals simultaneously. Then note that for any $\alpha < 1/2$, if $t \in [k, k+1)$ and $h < h(k)$ small enough,

$$|B(t+h) - B(t)| \leq C\sqrt{h \log(1/h)} \leq C h^{\alpha} (= C h^{1/2} (1/h)^{(1/2-\alpha)}).$$

This concludes the proof. ■

In fact:

THM 27.19 (Lévy's modulus of continuity) *Almost surely,*

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} = 1.$$

For the proof, see [MP10].

This result is tight. See [MP10, Remark 1.21].

3 Non-Monotonicity

A first example of "irregularity":

THM 27.20 *Almost surely, for all $0 < a < b < +\infty$, standard BM is not monotone on the interval $[a, b]$.*

Proof: It suffices to look at intervals with rational endpoints because any general non-degenerate interval of monotonicity must contain one of those. Since there are countably many rational intervals, it suffices to prove that any particular one has probability 0 of being monotone. Let $[a, b]$ be such an interval. Note that for any finite sub-division

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

the probability that each increment satisfies

$$B(a_i) - B(a_{i-1}) \geq 0, \qquad \forall i = 1, \ldots, n,$$

or the same with negative, is at most

$$2\left(\frac{1}{2}\right)^n \to 0,$$

as $n \to \infty$ by symmetry of Gaussians. ■

More generally, we can prove the following.

THM 27.21 *Almost surely, BM satisfies:*

1. *The set of times at which local maxima occur is dense.*
2. *Every local maximum is strict.*
3. *The set of local maxima is countable.*

Proof: Part (3). We use part (2). If t is a strict local maximum, it must be in the set

$$\bigcup_{n=1}^{+\infty} \left\{t \,:\, B(t,\omega) > B(s,\omega),\ \forall s,\ |s - t| < n^{-1}\right\}.$$

But for each n, the set must be countable because two such t's must be separated by n^{-1}. So the union is countable. ■

4 Non-differentiability

So $B(t)$ grows slower than t. But the following lemma shows that its limsup grows faster than $\sqrt{t}$.

LEM 27.22 *Almost surely*

$$\limsup_{n \to +\infty} \frac{B(n)}{\sqrt{n}} = +\infty.$$

And similarly for $\liminf$.

Proof: By reverse Fatou,

$$\mathbb{P}[B(n) > c\sqrt{n} \text{ i.o.}] \geq \limsup_{n\to+\infty} \mathbb{P}[B(n) > c\sqrt{n}] = \limsup_{n\to+\infty} \mathbb{P}[B(1) > c] > 0,$$

by the scaling property. Thinking of $B(n)$ as the sum of $X_n = B(n) - B(n-1)$, the event on the LHS is exchangeable and the Hewitt-Savage 0-1 law implies that it has probability 1 (where we used the positive lower bound). ∎

DEF 27.23 (Upper and lower derivatives) *For a function f, we define the* upper *and* lower right derivatives *as*

$$D^* f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h},$$

and

$$D_* f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

We begin with an easy first result.

THM 27.24 *Fix $t \geq 0$. Then almost surely Brownian motion is not differentiable at t. Moreover, $D^*B(t) = +\infty$ and $D_*B(t) = -\infty$.*

Proof: Consider the time inversion X. Then

$$D^*X(0) \geq \limsup_{n\to+\infty} \frac{X(n^{-1}) - X(0)}{n^{-1}} = \limsup_{n\to+\infty} B(n) = +\infty,$$

by the lemma above. This proves the result at 0. Then note that $X(s) = B(t+s) - B(s)$ is a standard Brownian motion and differentiability of X at 0 is equivalent to differentiability of B at t. ∎

In fact, we can prove something much stronger.

THM 27.25 *Almost surely, BM is nowhere differentiable. Furthermore, almost surely, for all t*

$$D^*B(t) = +\infty,$$

or

$$D_*B(t) = -\infty,$$

or both.

Proof: Suppose there is t_0 such that the latter does not hold. By boundedness of BM over $[0,1]$, we have

$$\sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M,$$

for some $M < +\infty$. Assume t_0 is in $[(k-1)2^{-n}, k2^{-n}]$ for some k, n. Then for all $1 \leq j \leq 2^n - k$, in particular, for $j = 1, 2, 3$,

$$\begin{aligned}
&|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \\
&\qquad \leq |B((k+j)2^{-n}) - B(t_0)| + |B(t_0) - B((k+j-1)2^{-n})| \\
&\qquad \leq M(2j+1)2^{-n},
\end{aligned}$$

by our assumption. Define the events

$$\Omega_{n,k} = \{|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n},\ j = 1,2,3\}.$$

It suffices to show that $\cup_{k=1}^{2^n-3}\Omega_{n,k}$ cannot happen for infinitely many n. Indeed,

$$\begin{aligned}
&\mathbb{P}\left[\exists t_0 \in [0,1],\ \sup_{h\in[0,1]} \frac{|B(t_0+h)-B(t_0)|}{h} \leq M\right] \\
&\qquad\qquad \leq \mathbb{P}\left[\bigcup_{k=1}^{2^n-3} \Omega_{n,k} \text{ for infinitely many } n\right].
\end{aligned}$$

(Then the result follows by taking all $[k, k+1]$ intervals and all M integers.) But by the independence of increments

$$\begin{aligned}
\mathbb{P}[\Omega_{n,k}] &= \prod_{j=1}^{3} \mathbb{P}[|B((k+j)2^{-n}) - B((k+j-1)2^{-n})| \leq M(2j+1)2^{-n}] \\
&\leq \mathbb{P}\left[|B(2^{-n})| \leq \frac{7M}{2^n}\right]^3 \\
&= \mathbb{P}\left[\left|\frac{1}{\sqrt{2^{-n}}} B\left(\left[\sqrt{2^{-n}}\right]^2\right)\right| \leq \frac{7M}{\sqrt{2^{-n}}\cdot 2^n}\right]^3 \\
&= \mathbb{P}\left[|B(1)| \leq \frac{7M}{\sqrt{2^n}}\right]^3 \\
&\leq \left(\frac{7M}{\sqrt{2^n}}\right)^3,
\end{aligned}$$

because the density of a standard Gaussian is bounded by $1/2$. (The choice of 3 comes from summability.) Hence

$$\mathbb{P}\left[\bigcup_{k=1}^{2^n-3}\Omega_{n,k}\right] \leq 2^n\left(\frac{7M}{\sqrt{2^n}}\right)^3 = (7M)^3 2^{-n/2},$$

which is summable. The result follows from BC. That is, the probability above is 0. ■

5 Quadratic variation

Recall:

DEF 27.26 (Bounded variation) *A function* $f : [0,t] \to \mathbb{R}$ *is of* bounded variation *if there is* $M < +\infty$ *such that*

$$\sum_{j=1}^{k}|f(t_j) - f(t_{j-1})| \leq M,$$

for all $k \geq 1$ *and all partitions* $0 = t_0 < t_1 < \cdots < t_k = t$. *Otherwise, we say that it is of* unbounded variation.

Functions of bounded variation are known to be differentiable. Since BM is nowhere differentiable, it must have unbounded variation. However, BM has a finite "quadratic variation."

THM 27.27 (Quadratic variation) *Suppose the sequence of partitions*

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = t,$$

is nested*, that is, at each step one or more partition points are added, and the* mesh

$$\Delta(n) = \sup_{1\leq j\leq k(n)}\{t_j^{(n)} - t_{j-1}^{(n)}\},$$

converges to 0*. Then, almost surely,*

$$\lim_{n\to+\infty}\sum_{j=1}^{k(n)}(B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t.$$

Proof: By considering subsequences, it suffices to consider the case where one point is added at each step. Let

$$X_{-n} = \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.$$

Let

$$\mathcal{G}_{-n} = \sigma(X_{-n}, X_{-n-1}, \ldots)$$

and

$$\mathcal{G}_{-\infty} = \bigcap_{k=1}^{\infty} \mathcal{G}_{-k}.$$

CLAIM 27.28 *We claim that $\{X_{-n}\}$ is a reversed MG.*

Proof: We want to show that

$$\mathbb{E}[X_{-n+1} \,|\, \mathcal{G}_{-n}] = X_{-n}.$$

In particular, this will imply by induction

$$X_{-n} = \mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-n}].$$

Assume that, at step n, the new point s is added between the old points $t_1 < t_2$. Write

$$X_{-n+1} = (B(t_2) - B(t_1))^2 + W,$$

and

$$X_{-n} = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2 + W,$$

where W is independent of the other terms. We claim that

$$\begin{aligned}\mathbb{E}[(B(t_2) - B(t_1))^2 \,|\, (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2] \\ = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2,\end{aligned}$$

which follows from the following lemma.

LEM 27.29 *Let $X, Z \in \mathcal{L}^2$ be independent and assume Z is symmetric. Then*

$$\mathbb{E}[(X + Z)^2 \,|\, X^2 + Z^2] = X^2 + Z^2.$$

Proof: By symmetry of Z,

$$\begin{aligned}\mathbb{E}[(X+Z)^2 \,|\, X^2+Z^2] &= \mathbb{E}[(X-Z)^2 \,|\, X^2+(-Z)^2] \\ &= \mathbb{E}[(X-Z)^2 \,|\, X^2+Z^2].\end{aligned}$$

Taking the difference we get

$$\mathbb{E}[XZ \,|\, X^2+Z^2] = 0.$$

■

The fact that X_{-n} is a reversed MG follows from the argument above. (Exercise.) ■

We return to the proof of the theorem. By Lévy's Downward Theorem,

$$X_{-n} \to \mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}],$$

almost surely. Note that $\mathbb{E}[X_{-1}] = \mathbb{E}[X_{-n}] = t$. Moreover, by (FATOU), the variance of the limit (the fourth central moment of the Gaussian in $3\sigma^4$)

$$\begin{aligned}\mathbb{E}[(\mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}] - t)^2] &\leq \liminf_n \mathbb{E}[(X_{-n}-t)^2] \\ &\leq \liminf_n \mathrm{Var}\left[\sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2\right] \\ &= \liminf_n 3 \sum_{j=1}^{k(n)} (t_j^{(n)} - t_{j-1}^{(n)})^2 \\ &\leq 3t \liminf_n \Delta(n) \\ &= 0.\end{aligned}$$

So finally

$$\mathbb{E}[X_{-1} \,|\, \mathcal{G}_{-\infty}] = t.$$

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.