Notes 26 : Brownian motion: definition

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References:[Dur10, Section 3.9, 8.1], [MP10, Section 1.1, Appendix B].

The goal of this lecture is to define and construct standard Brownian motion.

1 Multivariate Gaussians

We begin by reviewing some facts about multivariate Gaussians.

1.1 Random vectors

We first develop general tools to study multivariate distributions.

DEF 26.1 (Characteristic function) *The* characteristic function *(CF) of a random vector $X = (X_1, \ldots, X_d)$ is given by, for $t \in \mathbb{R}^d$,*

$$\phi_X(t) = \mathbb{E}\left[\exp\left(i(t_1 X_1 + \cdots + t_d X_d)\right)\right].$$

As in the one-dimensional case, we have an inversion formula:

THM 26.2 (Inversion formula) *Let μ be the probability measure corresponding to the random vector $(X_1, \ldots, X_d)$, that is, for all $B \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mu(B) = \mathbb{P}[(X_1, \ldots, X_d) \in B].$$

If $A = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $\mu(\partial A) = 0$ then

$$\mu(A) = \lim_{T \to +\infty} (2\pi)^{-d} \int_{[-T,T]^d} \prod_{j=1}^{d} \psi_j(t_j) \phi(t) \mathrm{d}t,$$

where

$$\psi_j(s) = \frac{\exp(-isa_j) - \exp(-isb_j)}{is}.$$

Proof: Follows from the one-dimensional inversion formula. See [Dur10, Theorem 3.9.3]. ■

An important application of the previous formula is:

THM 26.3 *The RVs $X_1, \ldots, X_d$ are independent if and only if*

$$\phi_X(t) = \prod_{j=1}^{d} \phi_{X_j}(t_j),$$

for all $t \in \mathbb{R}^d$ where $X = (X_1, \ldots, X_d)$.

Proof: The "only if" part is obvious. The inversion formula and Fubini's theorem gives the "if" part. ■

DEF 26.4 *A sequence of random vectors X_n converges weakly to X_∞, denoted $X_n \Rightarrow X_\infty$, if*

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)],$$

for all bounded continuous functions f. The portmanteau theorem gives equivalent characterizations.

In terms of CFs, we have:

THM 26.5 (Convergence theorem) *Let X_n, $1 \leq n \leq \infty$, be random vectors with CFs ϕ_n. A necessary and sufficient condition for $X_n \Rightarrow X_\infty$ is that*

$$\phi_n(t) \to \phi_\infty(t),$$

for all $t \in \mathbb{R}^d$.

Proof: Follows from the one-dimensional result. See [Dur10, Theorem 3.9.4]. ■

We require one last definition:

DEF 26.6 (Covariance) *Let $X = (X_1, \ldots, X_d)$ be a random vector with mean $\mu = \mathbb{E}[X]$. The* covariance of *X is the $d \times d$ matrix Γ with entries*

$$\Gamma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)].$$

1.2 Multivariate Gaussian: definition

Recall:

DEF 26.7 (Gaussian distribution) *A* standard Gaussian *is a RV Z with CF*

$$\phi_Z(t) = \exp\left(-t^2/2\right),$$

and density

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right).$$

In particular, Z has mean 0 and variance 1. More generally,

$$X = \sigma Z + \mu,$$

is a Gaussian RV with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

We will need a multivariate generalization of the standard Gaussian.

DEF 26.8 (Multivariate Gaussian) *A d-dimensional standard Gaussian is a random vector $X = (X_1, \ldots, X_d)$ where the X_is are independent standard Gaussians. In particular, X has mean 0 and covariance matrix I. More generally, a random vector $X = (X_1, \ldots, X_d)$ is Gaussian if there is a vector b, a $d \times r$ matrix A and an r-dimensional standard Gaussian Y such that*

$$X = AY + b.$$

Then X has mean $\mu = b$ and covariance matrix $\Gamma = AA^T$. The CF of X is given by

$$\phi_X(t) = \exp\left(i \sum_{j=1}^{d} t_j \mu_j - \frac{1}{2} \sum_{j,k=1}^{d} t_j t_k \Gamma_{jk}\right).$$

From the CF and the theorems above, we get the following:

COR 26.9 (Independence) *Let $X = (X_1, \ldots, X_d)$ be a multivariate Gaussian. Then the X_is are independent if and only if $\Gamma_{ij} = 0$ for all $i \neq j$, that is, if they are* uncorrelated.

COR 26.10 (Linear combinations) *The random vector $(X_1, \ldots, X_d)$ is multivariate Gaussian if and only if all linear combinations of its components are Gaussian.*

COR 26.11 (Convergence) *Let X_n be a sequence of multivariate Gaussian vectors with means μ_n and covariances Γ_n such that $X_n \to X_\infty$ a.s., $\mu_u \to \mu_\infty$, and $\Gamma_n \to \Gamma_\infty$. Then X_∞ is a multivariate Gaussian with mean μ_∞ and covariance matrix Γ_∞.*

Finally:

THM 26.12 (Multivariate CLT) *Let $X_1, X_2, \ldots$ be IID random vectors with means μ and finite covariance matrix Γ. Let $S_n = \sum_{j=1}^{n} X_j$, Then*

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow Z,$$

where Z is a multivariate Gaussian with mean 0 and covariance matrix Γ.

Proof: Follows easily from one-dimensional result. See [Dur10, Theorem 3.9.6]. ■

2 Brownian motion: definition

We give two equivalent definitions of Brownian motion. The first one relies on the notion of a Gaussian process.

DEF 26.13 (Gaussian process) *A continuous-time stochastic process* $\{X(t)\}_{t\geq 0}$ *is a* Gaussian process *if for all* $n \geq 1$ *and* $0 \leq t_1 < \cdots < t_n < +\infty$ *the random vector*

$$(X(t_1), \ldots, X(t_n)),$$

is multivariate Gaussian. The mean and covariance functions of X *are* $\mathbb{E}[X(t)]$ *and* $\mathrm{Cov}[X(s), X(t)]$ *respectively.*

DEF 26.14 (Brownian motion: Definition I) *The continuous-time stochastic process* $X = \{X(t)\}_{t\geq 0}$ *is a* standard Brownian motion *if* X *is a Gaussian process with almost surely continuous paths, that is,*

$$\mathbb{P}[X(t) \textit{ is continuous in } t] = 1,$$

such that $X(0) = 0$*,*

$$\mathbb{E}[X(t)] = 0,$$

and

$$\mathrm{Cov}[X(s), X(t)] = s \wedge t.$$

More generally, $B = \sigma X + x$ *is a* Brownian motion started at x.

From the properties of the multivariate Gaussian, we get the following equivalent definition. This one focuses on the properties of its increments.

DEF 26.15 (Stationary independent increments) *An SP* $\{X(t)\}_{t\geq 0}$ *has* stationary increments *if the distribution of* $X(t) - X(s)$ *depends only on* $t - s$ *for all* $0 \leq s \leq t$*. It has* independent increments *if the RVs* $\{X(t_{j+1} - X(t_j)), 1 \leq j < n\}$ *are independent whenever* $0 \leq t_1 < t_2 < \cdots < t_n$ *and* $n \geq 1$*.*

DEF 26.16 (Brownian motion: Definition II) *The continuous-time stochastic process* $X = \{X(t)\}_{t\geq 0}$ *is a* standard Brownian motion *if* X *has almost surely continuous paths and stationary independent increments such that* $X(s+t) - X(s)$ *is Gaussian with mean* 0 *and variance* t*.*

See [Dur10, Chapter 8.1] for proof of the equivalence.

3 Brownian motion: construction

Given that standard Brownian motion is defined in terms of finite-dimensional distributions, it is tempting to attempt to construct it by using Kolmogorov's Extension Theorem.

THM 26.17 (Kolmogorov's Extension Theorem: Uncountable Case) *Let*

$$\Omega_0 = \{\omega \,:\, [0,\infty) \to \mathbb{R}\},$$

and $\mathcal{F}_0$ be the σ-field generated by the finite-dimensional sets

$$\{\omega \,:\, \omega(t_i) \in A_i, 1 \leq i \leq n\},$$

for $A_i \in \mathcal{B}$. There is a unique probability measure ν on $(\Omega_0, \mathcal{F}_0)$ so that

$$\nu(\{\omega \,:\, \omega(0) = 0\}) = 1$$

and whenever $0 \leq t_1 < \cdots < t_n$ with $n \geq 1$ we have

$$\nu(\{\omega \,:\, \omega(t_i) \in A_i\}) = \mu_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n),$$

where the latter is the finite-dimensional distribution of standard Brownian motion.

See [Dur10]. The only problem with this approach is that the event

$$C = \{\omega \,:\, \omega(t) \text{ is continuous in } t\},$$

is not in $\mathcal{F}_0$. See Exercise 8.1.1 in [Dur10].

Instead, we proceed as follows. There are several constructions of Brownian motion. We present Lévy's contruction, as described in [MP10].

THM 26.18 (Existence) *Standard Brownian motion $B = \{B(t)\}_{t\geq 0}$ exists.*

Proof: We first construct B on $[0,1]$. The idea is to construct the process on dyadic points and extend it linearly. Let

$$\mathcal{D}_n = \{k2^{-n} \,:\, 0 \leq k \leq 2^n\},$$

and

$$\mathcal{D} = \cup_{n=0}^{\infty} \mathcal{D}_n.$$

Note that $\mathcal{D}$ is countable and consider $\{Z_t\}_{t\in\mathcal{D}}$ a collection of independent standard Gaussians. We define $B(d)$ for $d \in \mathcal{D}_n$ by induction. First take $B(0) = 0$

and $B(1) = Z_1$. Note that $B(1) - B(0)$ is Gaussian with variance 1. Then for $d \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}$ we let

$$B(d) = \frac{B(d-2^{-n}) + B(d+2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

By construction, $B(d)$ is independent of $\{Z_t \,:\, t \in \mathcal{D} \backslash \mathcal{D}_n\}$. Moreover, as a linear combination of zero-mean Gaussians, $B(d)$ is a zero-mean Gaussian.

We claim that the differences $B(d) - B(d - 2^{-n})$, for all $d \in \mathcal{D}_n \backslash \{0\}$, are independent Gaussians with variance 2^{-n} (recall that for Gaussians, pairwise independence suffices).

- We first argue about neighboring increments. Note that, for $d \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}$,

  $$B(d) - B(d-2^{-n}) = \frac{B(d+2^{-n}) - B(d-2^{-n})}{2} + \frac{Z_d}{2 \cdot 2^{(n-1)/2}},$$

  and

  $$B(d+2^{-n}) - B(d) = \frac{B(d+2^{-n}) - B(d-2^{-n})}{2} - \frac{Z_d}{2 \cdot 2^{(n-1)/2}},$$

  are Gaussians and they are independent by the following lemma. By induction the differences above are Gaussians with variance $2^{-(n-1)}$ and independent of Z_d.

  LEM 26.19 *If (X_1, X_2) is a standard Gaussian then so is $\frac{1}{\sqrt{2}}(X_1 + X_2, X_1 - X_2)$.*

- More generally, the two intervals are separated by $d \in \mathcal{D}_j$. Take a minimal such j. Then, by induction, the increments over the intervals $[d - 2^{-j}, d]$ and $[d, d + 2^{-j}]$ are independent. Moreover, the increments over the two intervals of length 2^{-n} of interest (included in the above intervals) are constructed from $B(d) - B(d - 2^{-j})$, respectively $B(d + 2^{-j}) - B(d)$, using a disjoint set of variables $\{Z_t \,:\, t \in \mathcal{D}_n\}$. That proves the claim by induction.

We now interpolate linearly between dyadic points. More precisely, let

$$F_0(t) = \begin{cases} Z_1, & t = 1, \\ 0, & t = 0, \\ \text{linearly}, & \text{in between.} \end{cases}$$

and for $n \geq 1$

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t, & t \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}, \\ 0, & t \in \mathcal{D}_{n-1}, \\ \text{linearly}, & \text{in between.} \end{cases}$$

We then have for $d \in \mathcal{D}_n$

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

Exercise: check by induction.

We want to show that the resulting process is continuous on $[0,1]$. We claim that the series

$$B(t) = \sum_{n=0}^{\infty} F_n(t),$$

is uniformly convergent. From a bound on Gaussian tails we saw last quarter,

$$\mathbb{P}[|Z_d| \geq c\sqrt{n}] \leq \exp\left(-c^2 n/2\right),$$

so that for c large enough

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}[\exists d \in \mathcal{D}_n, |Z_d| \geq c\sqrt{n}] &\leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(-c^2 n/2\right) \\ &< +\infty. \end{aligned}$$

By BC, there is N (random) such that $|Z_d| < c\sqrt{n}$ for all $d \in \mathcal{D}_n$ with $n > N$. In particular, for $n > N$ we have

$$\|F_n\|_\infty < c\sqrt{n} 2^{-(n+1)/2},$$

from which we get the claim.

To show that $B(t)$ has the correct finite-dimensional distributions, note that this is the case for $\mathcal{D}$ by the above argument. Since $\mathcal{D}$ is dense in $[0,1]$ the result holds on $[0,1]$ by taking limits and using the convergence theorem for Gaussians from the previous lecture.

Finally, we extend the process to $[0,+\infty)$ by gluing together independent copies of $B(t)$. ■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.