Notes 14 : Martingales

Math 733-734: Theory of Probability *Lecturer: Sebastien Roch*

References: [Wil91, Section 10], [Dur10, Section 5.2], [KT75, Section 6.1].

1 Martingales

1.1 Definitions

DEF 14.1 *A* filtered space *is a tuple* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ *where:*

- $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space*
- $\{\mathcal{F}_n\}$ *is a* filtration*, i.e.,*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_\infty \equiv \sigma(\cup \mathcal{F}_n) \subseteq \mathcal{F}.$$

where each $\mathcal{F}_i$ *is a* σ*-field.*

EX 14.2 *Let* $X_0, X_1, \ldots$ *be iid RVs. Then a filtration is given by*

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n), \; \forall n \geq 0.$$

Fix $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$.

DEF 14.3 *A process* $\{W_n\}_{n\geq 0}$ *is* adapted *if* $W_n \in \mathcal{F}_n$ *for all* n*.*

(Intuitively, the value of W_n is known at time n.)

EX 14.4 *Continuing. Let* $\{S_n\}_{n\geq 0}$ *where* $S_n = \sum_{i\leq n} X_i$ *is adapted.*

Our main definition is the following.

DEF 14.5 *A process* $\{M_n\}_{n\geq 0}$ *is a* martingale *(MG) if*

- $\{M_n\}$ *is adapted*
- $\mathbb{E}|M_n| < +\infty$ *for all* n
- $\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}$ *for all* $n \geq 1$

A superMG or subMG is similar but the last equality holds with $\leq$ *or* $\geq$ *respectively. (Note that for a MG, by (TOWER), we have* $\mathbb{E}[M_n \,|\, \mathcal{F}_m] = M_m$ *for all* $n > m$*.*

1.2 Examples

EX 14.6 (Sums of iid RVs with mean 0**)** *Let*

- $X_0, X_1, \ldots$ *iid RVs integrable and centered with* $X_0 = 0$
- $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$
- $S_n = \sum_{i \leq n} X_i$

Then note that $\mathbb{E}|S_n| < \infty$ *by triangle inequality and*

$$\begin{aligned}\mathbb{E}[S_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[S_{n-1} + X_n \,|\, \mathcal{F}_{n-1}] \\ &= S_{n-1} + \mathbb{E}[X_n] = S_{n-1}.\end{aligned}$$

EX 14.7 (Variance of a sum) *Same setup with* $\sigma^2 \equiv \mathrm{Var}[X_1] < \infty$*. Define*

$$M_n = S_n^2 - n\sigma^2.$$

Note that

$$\mathbb{E}|M_n| \leq \sum_{i \leq n} \mathrm{Var}[X_i] + n\sigma^2 \leq 2n\sigma^2 < +\infty$$

and

$$\begin{aligned}\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[(X_n + S_{n-1})^2 - n\sigma^2 \,|\, \mathcal{F}_{n-1}] \\ &= \mathbb{E}[X_n^2 + 2X_n S_{n-1} + S_{n-1}^2 - n\sigma^2 \,|\, \mathcal{F}_{n-1}] \\ &= \sigma^2 + 0 + S_{n-1}^2 - n\sigma^2 = M_{n-1}.\end{aligned}$$

EX 14.8 (Exponential moment of a sum; Wald's MG) *Same setup with* $\phi(\lambda) = \mathbb{E}[\exp(\lambda X_1)] < +\infty$ *for some* $\lambda \neq 0$*. Define*

$$M_n = \phi(\lambda)^{-n} \exp(\lambda S_n).$$

Note that

$$\mathbb{E}|M_n| \leq \frac{\phi(\lambda)^n}{\phi(\lambda)^n} = 1 < +\infty$$

and

$$\begin{aligned}\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \phi(\lambda)^{-n} \mathbb{E}[\exp(\lambda(X_n + S_{n-1})) \,|\, \mathcal{F}_{n-1}] \\ &= \phi(\lambda)^{-n} \exp(\lambda S_{n-1}) \phi(\lambda) = M_{n-1}.\end{aligned}$$

EX 14.9 (Product of iid RVs with mean 1**)** *Same setup with* $X_0 = 1$, $X_i \geq 0$ *and* $\mathbb{E}[X_1] = 1$. *Define*

$$M_n = \prod_{i \leq n} X_i.$$

Note that

$$\mathbb{E}|M_n| = 1$$

and

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}\mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}.$$

EX 14.10 (Accumulating data; Doob's MG) *Let* $X \in \mathcal{L}^1(\mathcal{F})$. *Define*

$$M_n = \mathbb{E}[X \,|\, \mathcal{F}_n].$$

Note that

$$\mathbb{E}|M_n| \leq \mathbb{E}|X| < +\infty,$$

and

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[X \,|\, \mathcal{F}_{n-1}] = M_{n-1},$$

by (TOWER).

EX 14.11 (Eigenvalues of transition matrix) *A Markov chain (MC) on a countable* E *is a process of the following form:*

- $\{\mu_i\}_{i \in E}$, $\{p(i,j)\}_{i,j \in E}$
- $Y(i,n) \sim p(i,\cdot)$ *(indep.)*
- $Z_0 \sim \mu$ *and* $Z_n = Y(Z_{n-1}, n)$.

Suppose $f : E \to \mathbb{R}$ *is s.t.*

$$\sum_j p(i,j) f(j) = \lambda f(i), \ \forall i,$$

with $\mathbb{E}|f(Z_n)| < +\infty$ *for all* n. *Define*

$$M_n = \lambda^{-n} f(Z_n).$$

Note that

$$\mathbb{E}|M_n| < +\infty,$$

and

$$\begin{aligned}\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \lambda^{-n}\mathbb{E}[f(Z_n) \,|\, \mathcal{F}_{n-1}] \\ &= \lambda^{-n} \sum_j p(Z_{n-1}, j) f(j) \\ &= \lambda^{-n} \cdot \lambda \cdot f(Z_{n-1}) = M_{n-1}.\end{aligned}$$

EX 14.12 (Branching Process) *A branching process is a process of the following form:*

- $X(i,n)$, $i \geq 1$ *and* $n \geq 1$, *iid with mean* m
- $Z_0 = 1$ *and* $Z_n = \sum_{i \leq Z_{n-1}} X(i,n)$

Note that for $f(j) = j$ *in the context of the previous example we have*

$$\sum_j p(i,j) j = mi,$$

so that $M_n = m^{-n} Z_n$ *is a MG.*

2 Connection to gambling

DEF 14.13 *A process* $\{C_n\}_{n\geq 1}$ *is* predictable *if* $C_n \in \mathcal{F}_{n-1}$ *for all* $n \geq 1$.

EX 14.14 *Continuing Example 14.2.* $C_n = \mathbb{1}\{S_{n-1} \leq k\}$ *is predictable.*

EX 14.15 *Let* $\{X_n\}_{n\geq 0}$ *be an integrable adapted process and* $\{C_n\}_{n\geq 1}$, *a bounded predictable process. Define*

$$M_n = \sum_{i \leq n} (X_i - \mathbb{E}[X_i \,|\, \mathcal{F}_{i-1}]) C_i.$$

Then

$$\mathbb{E}|M_n| \leq \sum_{i \leq n} 2\mathbb{E}|X_n| K < +\infty,$$

where $|C_n| < K$ *for all* $n \geq 1$, *and*

$$\begin{aligned}\mathbb{E}[M_n - M_{n-1} \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[(X_n - \mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}]) C_n \,|\, \mathcal{F}_{n-1}] \\ &= C_n(\mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}] - \mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}]) = 0.\end{aligned}$$

2.1 Fair games

Take the previous example with $\{X_n\}_{n\geq 0}$ a MG, that is,

$$M_n = (C \bullet X)_n \equiv \sum_{i\leq n} C_i(X_i - X_{i-1}),$$

where $\{(C \bullet X)_n\}_{n\geq 0}$ is called the *martingale transform* and is a discrete analogue of stochastic integration. If you think of $X_n - X_{n-1}$ as your net winnings per unit stake at time n, then C_n is a gambling strategy and $(C \bullet X)$ is your total winnings up to time n in a *fair game*.

Arguing as in the previous example, we have the following theorem.

THM 14.16 (You can't beat the system) *Let $\{C_n\}$ be a bounded predictable process and $\{X_n\}$ be a MG. Then $\{(C \bullet X)_n\}$ is also a MG. If, moreover, $\{C_n\}$ is nonnegative and $\{X_n\}$ is a superMG, then $\{(C \bullet X)_n\}$ is also a superMG.*

2.2 Stopping times

Recall:

DEF 14.17 *A random variable $T : \Omega \to \overline{\mathbb{Z}}_+ \equiv \{0, 1, \ldots, +\infty\}$ is called a* stopping time *if*

$$\{T \leq n\} \in \mathcal{F}_n, \ \forall n \in \overline{\mathbb{Z}}_+,$$

or, equivalently,

$$\{T = n\} \in \mathcal{F}_n, \ \forall n \in \overline{\mathbb{Z}}_+.$$

(To see the equivalence, note

$$\{T = n\} = \{T \leq n\} \setminus \{T \leq n-1\},$$

and

$$\{T \leq n\} = \cup_{i\leq n}\{T = i\}.)$$

In the gambling context, a stopping time is a time at which you decide to stop playing. That decision should only depend on the history up to time n.

EX 14.18 *Let $\{A_n\}$ be an adapted process and $B \in \mathcal{B}$. Then*

$$T = \inf\{n \geq 0 : A_n \in B\},$$

is a stopping time.

2.3 Stopped supermartingales are supermartingales

DEF 14.19 *Let $\{X_n\}$ be an adapted process and T be a stopping time. Then*

$$X_n^T(\omega) \equiv X_{T(\omega)\wedge n}(\omega),$$

is called $\{X_n\}$ stopped at T.

THM 14.20 *Let $\{X_n\}$ be a superMG and T be a stopping time. Then the stopped process X^T is a superMG and in particular*

$$\mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_0].$$

The same result holds at equality if $\{X_n\}$ is a MG.

Proof: Let

$$C_n^{(T)} = \mathbb{1}\{n \leq T\}.$$

Note that

$$\{C_n^{(T)} = 0\} = \{T \leq n-1\} \in \mathcal{F}_{n-1},$$

so that $C^{(T)}$ is predictable. It is also nonnegative and bounded. Note further that

$$(C^{(T)} \bullet X)_n = X_{T\wedge n} - X_0 = X_n^T - X_0.$$

Apply the previous theorem. ■

2.4 Optional stopping theorem

When can we say that $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$? As a counter-example, think of the simple random walk started at 0 with $T = \inf\{n \geq 0 \,:\, S_n = 1\}$, where $\mathbb{P}[T < +\infty] = 1$.

THM 14.21 *Let $\{X_n\}$ be a superMG and T be a stopping time. Then X_T is integrable and*

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_0].$$

if one of the following holds:

1. *T is bounded*
2. *X is bounded and T is a.s. finite*
3. *$\mathbb{E}[T] < +\infty$ and X has bounded increments*
4. *X is nonnegative and T is a.s. finite.*

The first three hold with equality if X is a MG.

Proof: From the previous theorem, we have

$$(*) \quad \mathbb{E}[X_{T\wedge n} - X_0] \leq 0.$$

1. Take $n = N$ in $(*)$ where $T \leq N$ a.s.
2. Take n to $+\infty$ and use (DOM).
3. Note that
$$|X_{T\wedge n} - X_0| \leq |\sum_{i\leq T\wedge n}(X_i - X_{i-1})| \leq KT,$$
where $|X_n - X_{n-1}| \leq K$ a.s. Use (DOM).
4. Use (FATOU).

■

3 Martingale convergence theorem

DEF 14.22 *We say that $\{X_n\}_n$ is* bounded in $\mathcal{L}^1$ *if*

$$\sup_n \mathbb{E}|X_n| < +\infty.$$

THM 14.23 (Martingale convergence theorem) *Let X be a superMG bounded in $\mathcal{L}^1$. Then X_n converges and is finite a.s. Moreover, let $X_\infty = \liminf_n X_n$ then $X_\infty \in \mathcal{F}_\infty$ and $\mathbb{E}|X_\infty| < +\infty$.*

To prove this key theorem, we use the connection to gambling.

3.1 A natural gambling strategy

Recall that

$$(C \bullet X)_n = \sum_{i\leq n} C_n(X_n - X_{n-1}),$$

where C_n is predictable and X_n is a superMG, can be interpreted as your net winnings in a game. A natural strategy is to choose $\alpha < \beta$ and apply the following

- REPEAT
  - Wait until X_n gets below α

  - Play a unit stake until X_n gets above β and stop playing

- UNTIL TIME N

More formally, let

$$C_1 = \mathbb{1}\{X_0 < \alpha\},$$

and

$$C_n = \mathbb{1}\{C_{n-1} = 1\}\mathbb{1}\{X_{n-1} \leq \beta\} + \mathbb{1}\{C_{n-1} = 0\}\mathbb{1}\{X_{n-1} < \alpha\}.$$

Then $\{C_n\}$ is predictable.

3.2 Upcrossings

Define the following stopping times. Let $T_0 = -1$,

$$T_{2k-1} = \inf\{n > T_{2k-2} \,:\, X_n < \alpha\},$$

and

$$T_{2k} = \inf\{n > T_{2k-1} \,:\, X_n > \beta\}.$$

The *number of upcrossings of* $[\alpha, \beta]$ *by time* N is

$$U_N[\alpha, \beta] = \sup\{k \,:\, T_{2k} \leq N\}.$$

LEM 14.24 (Doob's Upcrossing Lemma) *Let* $\{X_n\}$ *be a superMG. Then*

$$(\beta - \alpha)\mathbb{E}U_N[\alpha, \beta] \leq \mathbb{E}[(X_N - \alpha)^-].$$

Proof: Let $Y_n = (C \bullet X)_n$. Then $\{Y_n\}$ is a superMG and satisfies

$$Y_N \geq (\beta - \alpha)U_N[\alpha, \beta] - (X_N - \alpha)^-,$$

since $(X_N - \alpha)^-$ overestimates the loss during the last interval of play. The result follows from $\mathbb{E}[Y_N] \leq 0$. ■

COR 14.25 *Let* $\{X_n\}$ *be a superMG bounded in* $\mathcal{L}^1$*. Then*

$$U_N[\alpha, \beta] \uparrow U_\infty[\alpha, \beta],$$

$$(\beta - \alpha)\mathbb{E}U_\infty[\alpha, \beta] \leq |\alpha| + \sup_n \mathbb{E}|X_n| < +\infty,$$

so that

$$\mathbb{P}[U_\infty[\alpha, \beta] = \infty] = 0.$$

Proof: Use (MON). ■

3.3 Convergence theorem

We are ready to prove the main theorem.
Proof:(of THM 14.23) Let $\alpha < \beta \in \mathbb{Q}$ and

$$\Lambda_{\alpha,\beta} = \{\omega \,:\, \liminf X_n < \alpha < \beta < \limsup X_n\}.$$

Note that

$$\begin{aligned}\Lambda &\equiv \{\omega \,:\, X_n \text{ does not converge in } [-\infty, +\infty]\} \\ &= \{\omega \,:\, \liminf X_n < \limsup X_n\} \\ &= \cup_{\alpha<\beta\in\mathbb{Q}}\Lambda_{\alpha,\beta}.\end{aligned}$$

Since

$$\Lambda_{\alpha,\beta} \subseteq \{U_\infty[\alpha,\beta] = \infty\},$$

we have $\mathbb{P}[\Lambda_{\alpha,\beta}] = 0$. By countability, $\mathbb{P}[\Lambda] = 0$. Use (FATOU) on $|X_n|$ to conclude. ■

A very useful corollary:

COR 14.26 *If $\{X_n\}$ is a nonnegative superMG then X_n converges a.s.*

Proof: $\{X_n\}$ is bounded in $\mathcal{L}^1$ since

$$\mathbb{E}|X_n| = \mathbb{E}[X_n] \leq \mathbb{E}[X_0],\ \forall n.$$

■

EX 14.27 (Polya's Urn) *An urn contains 1 red ball and 1 green ball. At each time, we pick one ball and put it back with an extra ball of the same color. Let R_n (resp. G_n) be the number of red balls (resp. green balls) after the nth draw. Let $\mathcal{F}_n = \sigma(R_0, G_0, R_1, G_1, \ldots, R_n, G_n)$. Define M_n to be the fraction of green balls. Then*

$$\begin{aligned}\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \frac{R_{n-1}}{G_{n-1}+R_{n-1}}\frac{G_{n-1}}{G_{n-1}+R_{n-1}+1} \\ &\quad + \frac{G_{n-1}}{G_{n-1}+R_{n-1}}\frac{G_{n-1}+1}{G_{n-1}+R_{n-1}+1} \\ &= \frac{G_{n-1}}{G_{n-1}+R_{n-1}} = M_{n-1}.\end{aligned}$$

Since $M_n \geq 0$ and is a MG, we have $M_n \to M_\infty$ a.s. See [Dur10, Section 4.3] for distribution of the limit and a generalization, or decipher,

$$\mathbb{P}[G_n = m+1] = \binom{n}{m}\frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1},$$

so that

$$\mathbb{P}[M_n \le x] = \frac{\lfloor x(n+2) - 1 \rfloor}{n+1} \to x,$$

(by a sandwich argument).

EX 14.28 (Convergence in L^1?) *We give an example that shows that the conditions of the Martingale Convergence Theorem do not guarantee convergence of expectations. Let $\{S_n\}$ be SRW started at* 1 *and*

$$T = \inf\{n > 0 \,:\, S_n = 0\}.$$

Then $\{S_{T\wedge n}\}$ is a nonnegative MG. It can only converge to 0*. (Any other integer value would not be possible because convergence would have to have occurred at a finite time and the next time step would have to be different.) But $\mathbb{E}[X_0] = 1 \neq 0$.*

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010.

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[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.