Gronwall's Inequality

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Our purpose is to derive the usual Gronwall Inequality from the following

Abstract Gronwall Inequality

Let M be a topological space which also has a partial order which is sequentially closed in $M \times M$. Suppose that a map $\Gamma : M \to M$ preserves the order relation and has an attractive fixed point v. Then

$$u \le \Gamma(u) \implies u \le v.$$

Proof. Assume $u \le \Gamma(u)$. Since Γ preserves the order relation we get $u \le \Gamma^n(u)$ by induction. Since v is an attractive fixed point we have $v = \lim_{n\to\infty} \Gamma^n(u)$. Since the order relation is sequentially closed, we conclude $u \le v$ as required. ☐

Assume that the continuous functions $u, \kappa : [0, T] \to [0, \infty)$ and $K > 0$ satisfy

$$u(t) \le K + \int_0^t \kappa(s)u(s)\,ds$$

for all $t \in [0, T]$. Then the usual Gronwall inequality is

$$u(t) \le K \exp\left(\int_0^t \kappa(s)\,ds\right). \tag{1}$$

The usual proof is as follows. The hypothesis is

$$\frac{u(s)}{K + \displaystyle\int_0^s \kappa(r)u(r)\,dr} \le 1.$$

Multiply this by $\kappa(s)$ to get

$$\frac{d}{ds} \ln\left(K + \int_0^s \kappa(r)u(r)\,dr\right) \le \kappa(s)$$

Integrate from $s = 0$ to $s = t$, and exponentiate to obtain

$$K + \int_0^t \kappa(r)u(r)\,dr \le K \exp\left(\int_0^t \kappa(s)\,ds\right).$$

By hypothesis, the left side is $\geq u(t)$.

We now show how to derive the usual Gronwall inequality from the abstract Gronwall inequality. For $v : [0, T] \to [0, \infty)$ define $\Gamma(v)$ by

$$\Gamma(v)(t) = K + \int_0^t \kappa(s)v(s)\,ds. \tag{2}$$

In this notation, the hypothesis of Gronwall's inequality is $u \leq \Gamma(u)$ where $v \leq w$ means $v(t) \leq w(t)$ for all $t \in [0, T]$. Since $\kappa(t) \geq 0$ we have

$$v \leq w \implies \Gamma(v) \leq \Gamma(w).$$

Hence iterating the hypothesis of Gronwall's inequality gives

$$u \leq \Gamma^n(u).$$

Now change the dummy variable in (2) from s to s_1 and apply the inequality $u(s_1) \leq \Gamma(u)(s_1)$ to obtain

$$\Gamma^2(u)(t) = K + \int_0^t \kappa(s_1)K\,ds_1 + \int_0^t \int_0^{s_1} \kappa(s_1)\kappa(s_2)u(s_2)\,ds_2\,ds_1$$

More generally, by induction we have

$$\Gamma^n(u) = K \sum_{j=0}^{n-1} G_j(t) + E_n(t)$$

where

$$G_j(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{j-1}} \kappa(s_1) \cdots \kappa(s_j)\,ds_j \cdots ds_1$$

(with $G_0(t) = 1$) and

$$E_n(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \kappa(s_1) \cdots \kappa(s_n)u(s_n)\,ds_n \cdots ds_1$$

Now $G_j(t)$ is an integral over the j-simplex $0 \leq s_j \leq \cdots \leq s_1 \leq t$ and the integrand is symmetric under a permutation of the variables. Hence

$$G_j(t) = \frac{1}{j!} \int_0^t \int_0^t \cdots \int_0^t \kappa(s_1) \cdots \kappa(s_j)\,ds_j \cdots ds_1 = \frac{1}{j!} \left(\int_0^t \kappa(s)\,ds \right)^j.$$

Also $|E_n(t)|$ is bounded by an nth power times the area $1/n!$ of the n-simplex. Hence the term $E_n(t)$ converges uniformly to zero and the series limits to the series for the exponential function.

The above argument shows Γ has an attractive fixed point so we can also prove the Gronwall inequality by solving $v = \Gamma(v)$; the solution is

$$v(t) = K \exp\left(\int_0^t \kappa(s)\,ds \right).$$

We use this approach to prove a more general form of Gronwall's inequality where the constant K is replaced by a continuous function $K : [0, T] \to [0, \infty)$. Namely, assume that

$$u(t) \le K(t) + \int_0^t \kappa(s)u(s)\,ds \tag{3}$$

for all $t \in [0, T]$. We prove that

$$u(t) \le K(t) + \int_0^t \kappa(s)K(s)\exp\left(\int_s^t \kappa(r)\,dr\right)ds. \tag{4}$$

The abstract Gronwall inequality applies much as before so to prove (4) we show that the solution of

$$v(t) = K(t) + \int_0^t \kappa(s)v(s)\,ds \tag{5}$$

is

$$v(t) = K(t) + \int_0^t K(s)\kappa(s))\exp\left(\int_s^t \kappa(r)\,dr\right)ds \tag{6}$$

Equation (5) implies $\dot{v} = \dot{K} + \kappa v$. By variation of constants we seek a solution in the form

$$v(t) = C(t)\exp\left(\int_0^t \kappa(r)\,dr\right).$$

Plugging into $\dot{v} = \dot{K} + \kappa v$ gives

$$\dot{C}(t)\exp\left(\int_0^t \kappa(r)\,dr\right) = \dot{K}(t)$$

so

$$C(t) = C(0) + \int_0^t \dot{K}(s)\exp\left(-\int_0^s \kappa(r)\,dr\right)ds$$

so

$$v(t) = C(0)\exp\left(\int_0^t \kappa(r)\,dr\right) + \int_0^t \dot{K}(s)\exp\left(\int_s^t \kappa(r)\,dr\right)ds$$

Equation (5) requires $v(0) = K(0)$ so

$$C(0) = K(0).$$

Integration by parts gives

$$\int_0^t \dot{K}(s)\exp\left(\int_s^t \kappa(r)\,dr\right)ds =$$
$$K(t) - K(0)\exp\left(\int_0^t \kappa(r)\,dr\right) + \int_0^t K(s)\kappa(s)\exp\left(\int_s^t \kappa(r)\,dr\right)ds$$

Combining the last three displayed equations give (6).

Here is the proof of (4) sketched in Exercise 1 Chapter 1 of [1]. Define

$$R(t) := \int_0^t \kappa(r)u(r)\,dr.$$

Then the derivative R' satisfies

$$R'(s) - k(s)R(s) = \kappa(s)\big(u(s) - R(s)\big) \leq \kappa(s)K(s).$$

Hence

$$\frac{d}{ds}\,R(s)\exp\left(\int_s^t \kappa(r)\,dr\right) \leq \kappa(s)K(s)\exp\left(\int_s^t \kappa(r)\,dr\right)$$

so integrating gives

$$R(t) = R(t) - R(0) \leq \int_0^t \kappa(s)K(s)\exp\left(\int_s^t \kappa(r)\,dr\right)\,ds.$$

Now add $K(t)$ to both sides and use the hypothesis $u(t) \leq K(t) + R(t)$.

If $K(t)$ is a constant, the right hand side of (4) reduces to the right hand side of (1). This follows on taking $K(t)$ constant in the fixed point equation $v = K + \int \kappa v$, but here's a direct proof.

$$\begin{aligned}
&K + \int_0^t \kappa(s)K\exp\left(\int_s^t \kappa(r)\,dr\right)\,ds \\
&\quad= K - K\int_0^t \frac{d}{ds}\exp\left(\int_s^t \kappa(r)\,dr\right)\,ds \\
&\quad= K - K\left(\exp(0) - \exp\left(\int_0^t \kappa(r)\,dr\right)\right) \\
&\quad= K\exp\left(\int_0^t \kappa(r)\,dr\right).
\end{aligned}$$

References

[1] E. Coddington & N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955.