Instability of an anisotropic micropolar fluid

Antoine Remond-Tiedrez

University of Wisconsin, Madison

Online North East PDE and Analysis Seminar (May 2020)
Collaborator

Ian Tice
Carnegie Mellon University
Part 1

What is a micropolar fluid?
Spot the difference resemblance

(a) Blood  (b) Sperm  (c) Liquid crystal

(d) Ferrofluid  (e) Knee joint  (f) Polymer melt

Antoine Remond-Tiedrez  (Wisconsin)  Anisotropic micropolar fluids  May 2020
Micropolar fluids and continuum mechanics

Complete kinematic description: flow map $\eta$. Initial micro-inertia ($J_0$). And micro-rotation map $Q$.

Aside - The motion of rigid bodies

Complete kinematic description:

- **translation** of the center of mass $\mathbf{x}(0) \mapsto \mathbf{x}(t)$ and
- **rotation** about the center of mass: $R(t) \in SO(3)$.

Complete dynamical description:

- **velocity** $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ and
- **angular velocity tensor** $\Theta = \frac{dR}{dt} R^T$.

**Note:** The matrix $\Theta$ is antisymmetric. When $d = 3$ we associate to $\Theta$ the vector $\theta = \text{vec} \Theta$ via $\Theta \mathbf{v} = \theta \times \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^3$. 
Aside - The **physics** of rigid bodies

The physics of the rigid body are fully described by

- its **mass** $m$ (response to **forces**)
- its **moment of inertia** $J$ (response to **torques**),

$$J = \int |x - \bar{x}|^2 - (x - \bar{x}) \otimes (x - \bar{x}) \, d\mu(x).$$

**Note:** the spectrum of $J$ is preserved by rigid motions.

Other physical quantities can then be computed.

- Linear momentum: $mv$.
- Angular momentum: $J\theta$.
- Kinetic energy: $\frac{1}{2} m|v|^2 + \frac{1}{2} J\theta \cdot \theta$. 
Micropolar fluids and continuum mechanics

How to derive the (incompressible) equations of micropolar fluids

1. Conservation laws
   • Mass: $$(\partial_t + u \cdot \nabla)\rho = 0.$$  
   • Linear momentum: $$(\partial_t + u \cdot \nabla)(\rho u) = \nabla \cdot T.$$  
   • Angular momentum: $T$ is symmetric. $$(\partial_t + u \cdot \nabla)(J\omega)J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega = 2\text{vec } T + \nabla \cdot M.$$  
   • Micro-inertia: $$(\partial_t + u \cdot \nabla)J = [\Omega, J],$$ where $\omega = \text{vec } \Omega.$

2. Linear stress-strain relation
   • Stress $T$ linear in $$(\mathbb{D}u, \frac{1}{2} \nabla \times u - \omega)$$  
   • Couple-stress $M$ linear in $\nabla \omega$

3. Frame-invariance

$$(1) - (3) \Rightarrow \text{incompressible micropolar fluids}$$
The equations of micropolar fluids

The equations are

\[
\begin{align*}
\rho(\partial_t + u \cdot \nabla)u &= \tilde{\mu}\Delta u + \kappa\nabla \times \omega - \nabla p + \mathcal{F} \quad \text{on } \mathbb{T}^3, \\
\nabla \cdot u &= 0 \quad \text{on } \mathbb{T}^3 \\
 J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega &= \kappa\nabla \times u - 2\kappa\omega + (\tilde{\alpha} - \tilde{\gamma})\nabla(\nabla \cdot \omega) + \tilde{\gamma}\Delta \omega + \mathcal{T} \quad \text{on } \mathbb{T}^3, \quad \text{and} \\
(\partial_t + u \cdot \nabla)J &= [\Omega, J] \quad \text{on } \mathbb{T}^3,
\end{align*}
\]

where \(\tilde{\mu}, \kappa, \tilde{\alpha}, \tilde{\gamma} > 0\) are viscosity constants and \(\mathcal{F}\) and \(\mathcal{T}\) correspond to external forces and torques, respectively. This is supplemented by initial conditions \((u_0, p_0, \omega_0, J_0)\).
Background and known results

- Introduced by Eringen ('66) amongst a hierarchy of models, c.f. his two-part textbook ('99, '01).

\[\text{micropolar} \subseteq \text{micro-stretch} \subseteq \text{micromorphic}\]
\[(\text{rigid}) \quad (\text{linear elasticity}) \quad (\text{nonlinear elasticity})\]

- The Cosserats were considering internal couples in 1909!
- Mathematically, only isotropic \((J \propto I)\) micropolar fluids have been studied.
  - 2D: globally well-posed (Łukaszewicz '01).
  - 3D: weak solutions locally-in-time, strong solutions are unique (Łukasszewicz '90 and '89, resp.).
  - Results on “partially inviscid” limits (see for example Dong-Zhang '10).
Part 2

Our result.
Our setup

- The microstructure is **anisotropic** but has an **axis of symmetry**.

- There is a **constant micro-torque** acting on the microstructure.
The PDE

In this context, the equations are

\[
\begin{align*}
\rho (\partial_t + u \cdot \nabla) u &= \tilde{\mu} \Delta u + \kappa \nabla \times \omega - \nabla p \text{ on } \mathbb{T}^3, \\
\nabla \cdot u &= 0 \text{ on } \mathbb{T}^3, \\
J (\partial_t + u \cdot \nabla) \omega + \omega \times J \omega &= \tau e_3 \\
&\quad + \kappa \nabla \times \omega - 2\kappa \omega + (\tilde{\alpha} - \tilde{\gamma}) \nabla (\nabla \cdot \omega) + \tilde{\gamma} \Delta \omega \text{ on } \mathbb{T}^3, \text{ and} \\
(\partial_t + u \cdot \nabla) J &= [\Omega, J] \text{ on } \mathbb{T}^3,
\end{align*}
\]

supplemented by initial conditions \((u_0, p_0, \omega_0, J_0)\), where \(\tau > 0\) is the magnitude of the external micro-torque.

Axially symmetric microstructure:
\[
\sigma(J_0) = \sigma(J) = \{\lambda, \lambda, \nu\} \text{ for some } \lambda, \nu > 0.
\]
In the presence of a constant micro-torque and provided the microstructure has an axis of symmetry, the system has a unique equilibrium.

**Theorem**

- If the microstructure is rod-like then the equilibrium is nonlinearly unstable in $L^2$.
- If the microstructure is pancake-like then the equilibrium is nonlinearly stable in $H^s$ with algebraic decay to equilibrium.
Part 3

The difficulties,
or how anisotropy makes things interesting.
Precession

Micropolar fluid:

\[ \mathcal{J} (\partial_t + u \cdot \nabla) \omega + \omega \times \mathcal{J} \omega \]

\[ = \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega. \]

Freely rotating rigid body with inertia \( \mathcal{J} \) and angular velocity \( \theta \):

\[ \frac{d}{dt} (\mathcal{J} \theta) = \mathcal{J} \frac{d}{dt} \theta + \theta \times \mathcal{J} \theta = 0. \]

Recall: \( \frac{d}{dt} \mathcal{J} = [\Theta, \mathcal{J}] \), where \( \Theta = \text{vec} \theta \).
The unstable case
Unstable case - The role of anisotropy

Strategy

• Find the **fastest** growing mode of the linearization about equilibrium.
• Prove that this growing mode is “**nonlinearly stable**”.

Difficulty: Precession

Due to **precession**, the linearization is **not** self-adjoint. Recall:

\[
J \left( \partial_t \omega + (u \cdot \nabla) \omega \right) + \omega \times J \omega
= \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega.
\]
Unstable case - Spectral analysis

Calling the linearized operator \( \mathcal{L} \), we study the spectrum of \( \hat{\mathcal{L}}(k) \) for \( k \in \mathbb{Z}^3 \). whose characteristic polynomial is \( p(z; k) \).

Note: \( p(z; k) = p(z; k, \tau, \kappa, \ldots) \) is of degree 8.

\[
p \approx p_0 + p_1 \text{ as } |k| \to \infty.
\]

There is no spectral gap.

\[
\hat{\mathcal{L}} = S + A
\]
Unstable case - Result

Theorem (Tice and R.T., 2019)

Suppose that the micro-structure is rod-like, i.e. $\lambda > \nu$, and that

$$X_{eq} = (u_{eq}, \omega_{eq}, J_{eq}) = \left(0, \frac{T}{2\kappa} e_3, \text{diag}(\lambda, \lambda, \nu)\right)$$

is the equilibrium solution. There exist constants $\sigma, \delta > 0$ such that for every $0 < \varepsilon < \delta$, there exists a strong solution $X = (u, \omega, J) \in L^\infty H^4$ on the time scale $T_\varepsilon \sim \log \frac{1}{\varepsilon}$ such that

$$\|X(0) - X_{eq}\|_{L^2} < \varepsilon \quad \text{and} \quad \|X(T_\varepsilon) - X_{eq}\|_{L^2} > \sigma.$$ 

The role of anisotropy

Anisotropy leads to precession, precession leads to not being self-adjoint, not being self-adjoint leads to suffering.
The stable case
Stable case

Strategy – “Control everything”

(1) Linear analysis: what rate of decay can we expect?
(2) Nonlinear analysis: what estimates do we have?
(3) Nonlinear analysis: can they control the interactions?
Stable case – Linear analysis

Consider, where \( a = (J_{13}, J_{23}) \) and \( \chi > 0 \), this cartoon of the linearization

\[
\begin{align*}
\partial_t \omega &= -\omega + \Delta \omega + a \\
\partial_t a &= -\chi \omega
\end{align*}
\]

Solutions satisfy the energy estimate

\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |\omega|^2 + \int_{T^3} \frac{1}{2} \chi |a|^2 = - \int_{T^3} |\omega|^2 + |\nabla \omega|^2.
\]

Bootstrapping (formally):

\[
\mathcal{E} = ||\omega||_{L^2}^2 + ||a||_{L^2}^2 + ||\partial_t \omega||_{L^2}^2 + ||\partial_t a||_{L^2}^2 \\
\mathcal{D} = ||\omega||_{H^1}^2 + ||\partial_t \omega||_{H^1}^2 \gtrsim ||a||_{H^{-1}}^2 + ||\partial_t a||_{H^1}^2
\]
Stable case – Linear analysis: hypo-coercivity

Recall:

\[ E = ||\omega||^2_{L^2} + ||a||^2_{L^2} + ||\partial_t \omega||^2_{L^2} + ||\partial_t a||^2_{L^2} \]
\[ D = ||\omega||^2_{H^1} + ||\partial_t \omega||^2_{H^1} \gtrsim ||a||^2_{H^{-1}} + ||\partial_t a||^2_{H^1} \]

Hypo-coercivity

Goal: \( E \lesssim D^\theta \) for some \( \theta \in (0, 1) \).

By interpolation (formally):

\[ ||a||^2_{L^2} \lesssim ||a||^2_{H^{-1}} ||a||^{2(1-\theta)}_{H^s} \]

where \( \theta = \frac{s}{1+s} \uparrow 1 \) as \( s \uparrow \infty \). Therefore

\[ E \lesssim D^\theta E_{\text{high}}^{1-\theta} \Rightarrow E(t) \lesssim \frac{E(0)}{(1+t)^\alpha}, \quad \alpha = \frac{\theta}{1-\theta} \uparrow \infty \text{ as } \theta \uparrow 1. \]

Antoine Remond-Tiedrez  (Wisconsin)  Anisotropic micropolar fluids  May 2020
Stable case – Energy estimates

Recall: the equation for $\partial_t \omega$ is

\[
J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega = \tau e_3 + \kappa \nabla \times u - 2\kappa\omega + (\tilde{\alpha} - \tilde{\gamma}) \nabla (\nabla \cdot \omega) + \tilde{\gamma} \Delta \omega.
\]

Question: for $\theta \sim \partial^\alpha \omega$, which terms do we keep?

A: Because of the precession, we keep an “advection-rotation” operator. If $\theta$ solves

\[
J(\partial_t + u \cdot \nabla)\theta + (\omega \times J)\theta = C
\]

subject to $\nabla \cdot u = 0$ and $(\partial_t + u \cdot \nabla) J = [\Omega, J]$, where $\omega = \text{vec} \, \Omega$, then

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} J\theta \cdot \theta = \int_{\mathbb{T}^3} C \cdot \theta.
\]
Stable case – Energy estimates (continued)

For the full system involving \((u, \omega, J)\) and including dissipation, the energy estimate we obtain is (neglecting commutators),

\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} \rho |u|^2 + \frac{1}{2} J \omega \cdot \omega + \frac{C}{(\nu - \lambda)} |a|^2 \\
= - \int_{T^3} \frac{\mu}{2} |\mathbb{D}u|^2 + 2\kappa \left| \frac{1}{2} \nabla \times u - \omega \right|^2 + \alpha |\nabla \cdot \omega|^2 + \frac{\beta}{2} |\mathbb{D}^0 \omega|^2 + 2\gamma |\nabla \times \omega|^2.
\]

where \(a = (J_{12}, J_{23})\).

This gives us control over \(H^k\) norms of \((u, \omega, a)\).
Stable case – Transport estimates

If $u$ is divergence-free and $J$ solves the advection-rotation equation

$$(\partial_t + u \cdot \nabla - [\Omega, \cdot ]) J = M \text{ on } \mathbb{T}^3,$$

where $M$ is a symmetric matrix field, then

$$\frac{d}{dt} \| J \|_{L^p(\mathbb{T}^3)} \leq \| M \|_{L^p(\mathbb{T}^3)}.$$

Key: $[A, S] : S = 0$ for any skew-symmetric $A$ and symmetric $S$.

Combined with high-low estimates, this gives us control over $H^k$ norms of $J$ (in a small energy regime):

$$\| J(t) \|_{H^k} \lesssim \| J_0 \|_{H^k} + \int_0^t \| (u, \omega)(s) \|_{H^k} ds.$$
Stable case – Controlling the interactions

Combining the energy and transport estimates and taking into account the hypo-coercivity the following picture emerges.

<table>
<thead>
<tr>
<th></th>
<th>low regularity</th>
<th>high regularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(u, \omega, a)$</td>
<td>decaying</td>
<td>bounded</td>
</tr>
<tr>
<td>$J$</td>
<td>bounded</td>
<td>growing</td>
</tr>
</tbody>
</table>

Interactions – goal: $|I| \lesssim \sqrt{\mathcal{E}\mathcal{D}}$ (because $\mathcal{E}' + \mathcal{D} = I$).

$$I = \int_{\mathbb{T}^3} (\partial_x^\alpha J) \omega \cdot \partial_x^\alpha a \sim \int_{\mathbb{T}^3} (\partial_x^{\alpha+1} J) \omega \cdot \partial_x^{\alpha-1} a$$

Only holds in a time-integrated fashion, i.e. $\int_0^t |I| \lesssim \mathcal{E}_0 \int_0^t \mathcal{D}$. 
Stable case – Result

Suppose that the micro-structure is pancake-like, i.e. $\nu > \lambda$, and that

$$X_{eq} = (u_{eq}, \omega_{eq}, a_{eq}) = \left(0, \frac{\tau}{2\kappa} e_3, \text{diag} (\lambda, \lambda, \nu)\right)$$

is the equilibrium solution. For any $k \geq 8$ there exists $\delta > 0$ such that the equations are globally well-posed in a $\delta$-ball in $H^k$ about $X_{eq}$. Moreover $(u, \omega, a)$ approach equilibrium at a rate $(1 + t)^{k-s}$ in $H^s$ (for $0 \leq s < k$).

The role of anisotropy

Anisotropy leads to

- hypo-coercivity at the linear level,
- estimates involving advection-rotation operators, and
- hyperbolic-parabolic interaction at the nonlinear level.
Conclusion
Conclusion

- Rigid microstructure + continuum mechanics = micropolar fluids.
- Subject to a uniform micro-torque, we obtain a sharp nonlinear stability criterion for axially symmetric microstructure.

![Diagram showing rod-like and pancake-like structures]

- Rod-like: unstable
- Pancake-like: stable

- Anisotropy leads to several difficulties:

<table>
<thead>
<tr>
<th></th>
<th>unstable case</th>
<th>stable case</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear level</td>
<td>not self-adjoint</td>
<td>hypo-coercivity</td>
</tr>
<tr>
<td>nonlinear level</td>
<td>no spectral gap</td>
<td>parabolic-hyperbolic interactions</td>
</tr>
</tbody>
</table>
Thank you for your attention!