Instability of an anisotropic micropolar fluid

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Part 1

What is a micropolar fluid?
Spot the difference resemblance

(a) Blood  (b) Sperm  (c) Liquid crystal

(d) Ferrofluid  (e) Knee joint  (f) Polymer melt
Micropolar fluids and continuum mechanics

Complete kinematic description: flow map $\eta$. initial micro-inertia ($J_0$). and micro-rotation map $Q$.

Unknowns: velocity $u$. micro-inertia $J$, and angular velocity $\omega$. 

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Anisotropic micropolar fluids  
May 2022
Aside - The motion of rigid bodies

Complete kinematic description:
- translation of the center of mass $\vec{x}(0) \mapsto \vec{x}(t)$ and
- rotation about the center of mass: $R(t) \in SO(3)$.

Complete dynamical description:
- velocity $v = \frac{d\vec{x}}{dt}$ and
- angular velocity tensor $\Theta = \frac{dR}{dt} R^T$.

Note: The matrix $\Theta$ is antisymmetric. When $d = 3$ we associate to $\Theta$ the vector $\theta = \text{vec} \Theta$ via $\Theta v = \theta \times v$ for every $v \in \mathbb{R}^3$. 
Aside - The physics of rigid bodies

The physics of the rigid body are fully described by

- its mass \( m \) (response to forces) and
- its moment of inertia \( J \) (response to torques),

\[
J = \int |x - \bar{x}|^2 I - (x - \bar{x}) \otimes (x - \bar{x}) d\mu(x).
\]

**Note:** the spectrum of \( J \) is preserved by rigid motions.

Other physical quantities can then be computed.

- Linear momentum: \( mv \).
- Angular momentum: \( J \theta \).
- Kinetic energy: \( \frac{1}{2} m|v|^2 + \frac{1}{2} J \theta \cdot \theta \).
Micropolar fluids and continuum mechanics

How to derive the (incompressible) equations of micropolar fluids

1. Conservation laws
   - Mass: \((\partial_t + u \cdot \nabla)\rho = 0\).
   - Linear momentum: \((\partial_t + u \cdot \nabla)(\rho u) = \nabla \cdot T\).
   - Angular momentum:
     \[ J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega = 2\text{vec } T + \nabla \cdot M. \]
   - Micro-inertia: \((\partial_t + u \cdot \nabla)J = [\Omega, J] \), where \(\omega = \text{vec } \Omega\).

2. Linear stress-strain relation
   - Stress \( T \) linear in \((\mathbb{D}u, \frac{1}{2} \nabla \times u - \omega)\)
   - Couple-stress \( M \) linear in \(\nabla \omega\)

3. Frame-invariance

\[(1) - (3) \Rightarrow \text{(incompressible) Navier-Stokes}\]
The equations of micropolar fluids

The equations are

\[
\begin{cases}
\rho\left(\partial_t + u \cdot \nabla\right)u = \tilde{\mu} \Delta u + \kappa \nabla \times \omega - \nabla p + \mathcal{F} \text{ on } \mathbb{T}^3, \\
\nabla \cdot u = 0 \text{ on } \mathbb{T}^3 \\
J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega \\
= \kappa \nabla \times u - 2\kappa \omega + (\tilde{\alpha} - \tilde{\gamma}) \nabla (\nabla \cdot \omega) + \tilde{\gamma} \Delta \omega + \mathcal{T} \text{ on } \mathbb{T}^3, \text{ and} \\
(\partial_t + u \cdot \nabla)J = [\Omega, J] \text{ on } \mathbb{T}^3,
\end{cases}
\]

where $\tilde{\mu}, \kappa, \tilde{\alpha}, \tilde{\gamma} > 0$ are viscosity constants and $\mathcal{F}$ and $\mathcal{T}$ correspond to external forces and torques, respectively. This is supplemented by initial conditions $(u_0, p_0, \omega_0, J_0)$. 
Background and known results

- Introduced by Eringen ('66) amongst a hierarchy of models, c.f. his two-part textbook ('99, ’01).

\[
\text{micropolar} \subseteq \text{micro-stretch} \subseteq \text{micromorphic} \\
\text{(rigid)} \quad \text{(linear elasticity)} \quad \text{(nonlinear elasticity)}
\]

- The Cosserats were considering internal couples in 1909!
- Mathematically, only isotropic \((J \propto I)\) micropolar fluids have been studied.
  - 2D: globally well-posed (Łukaszewicz ’01).
  - 3D: weak solutions locally-in-time, strong solutions are unique (Łukasszewicz ’90 and ’89, resp.).
  - Results on “partially inviscid” limits (see for example Dong-Zhang ’10).
Part 2

Our result.
Our setup

- The microstructure is anisotropic but has an axis of symmetry.

Isotropic  Rod-like  Pancake-like  “Fully anisotropic”

Axis of symmetry

- There is a constant micro-torque acting on the microstructure.
The PDE

In this context, the equations are

\[
\begin{cases}
\rho(\partial_t + u \cdot \nabla) u = \tilde{\mu} \Delta u + \kappa \nabla \times \omega - \nabla p \text{ on } \mathbb{T}^3, \\
\nabla \cdot u = 0 \text{ on } \mathbb{T}^3, \\
J(\partial_t + u \cdot \nabla) \omega + \omega \times J \omega = \tau e_3 \\
+ \kappa \nabla \times \omega - 2\kappa \omega + (\tilde{\alpha} - \tilde{\gamma}) \nabla (\nabla \cdot \omega) + \tilde{\gamma} \Delta \omega \text{ on } \mathbb{T}^3, \text{ and} \\
(\partial_t + u \cdot \nabla) J = [\Omega, J] \text{ on } \mathbb{T}^3,
\end{cases}
\]

supplemented by initial conditions \((u_0, p_0, \omega_0, J_0)\), where \(\tau > 0\) is the magnitude of the external micro-torque.

Axially symmetric microstructure:
\[
\sigma(J_0) = \sigma(J) = \{\lambda, \lambda, \nu\} \text{ for some } \lambda, \nu > 0.
\]
Result

In the presence of a constant micro-torque and provided the microstructure has an axis of symmetry, the system has a unique equilibrium.

![Diagram showing rod-like and pancake-like microstructures]

Theorem (Tice and R.T., 2019–20)

- If the microstructure is rod-like then the equilibrium is nonlinearly unstable in $L^2$.
- If the microstructure is pancake-like then the equilibrium is nonlinearly stable in $H^s$ with algebraic decay to equilibrium.
Part 3

The difficulties,
or how anisotropy makes things interesting.
Precession

Micropolar fluid:

\[ J \left( \partial_t + u \cdot \nabla \right) \omega + \omega \times J \omega = \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega. \]

Freely rotating rigid body with inertia \( J \) and angular velocity \( \theta \):

\[ \frac{d}{dt} (J \theta) = J \frac{d}{dt} \theta + \theta \times J \theta = 0. \]

Recall: \( \frac{d}{dt} J = [\Theta, J] \), where \( \theta = \text{vec} \theta \).
The unstable case
Unstable case - The role of anisotropy

Strategy

- Find the fastest growing mode of the linearization about equilibrium.
- Prove that this growing mode is “nonlinearly stable”.

Difficulty: Precession

Due to precession, the linearization is not self-adjoint. Recall:

\[
J \left( \partial_t \omega + (u \cdot \nabla) \omega \right) + \omega \times J \omega \\
= \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega.
\]
Unstable case - Spectral analysis

Calling the linearized operator $\mathcal{L}$, we study the spectrum of $\hat{\mathcal{L}}(k)$ for $k \in \mathbb{Z}^3$. whose characteristic polynomial is $p(z; k)$. 

Note: $p(z; k) = p(z; k, \tau, \kappa, \ldots)$ is of degree 8.

$$p \approx p_0 + p_1 \text{ as } |k| \to \infty.$$ 

There is no spectral gap.

$$\hat{\mathcal{L}} = S + A$$
Unstable case - Result

**Theorem (Tice and R.T., 2019)**

*Suppose that the micro-structure is rod-like, i.e. \( \lambda > \nu \), and that*

\[
X_{eq} = (u_{eq}, \omega_{eq}, J_{eq}) = \left(0, \frac{T}{2\kappa} e_3, \text{diag}(\lambda, \lambda, \nu)\right)
\]

*is the equilibrium solution. There exist constants \( \sigma, \delta > 0 \) such that for every \( 0 < \varepsilon < \delta \), there exists a strong solution \( X = (u, \omega, J) \in L^\infty H^4 \) on the time scale \( T_\varepsilon \sim \log \frac{1}{\varepsilon} \) such that*

\[
\|X(0) - X_{eq}\|_{L^2} < \varepsilon \quad \text{and} \quad \|X(T_\varepsilon) - X_{eq}\|_{L^2} > \sigma.
\]

**The role of anisotropy**

Anisotropy leads to precession, precession leads to not being self-adjoint, not being self-adjoint leads to suffering.
The stable case
Stable case

Strategy – “Control everything”

1. Linear analysis: what rate of decay can we expect?
2. Nonlinear analysis: what estimates do we have?
3. Nonlinear analysis: can they control the interactions?
Stable case – Linear analysis

Consider, where \( a = (J_{13}, J_{23}) \) and \( \chi > 0 \), this cartoon of the linearization

\[
\begin{align*}
\partial_t \omega &= -\omega + \Delta \omega + a \\
\partial_t a &= -\chi \omega
\end{align*}
\]

Solutions satisfy the energy estimate

\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |\omega|^2 + \int_{T^3} \frac{1}{2} \chi |a|^2 = - \int_{T^3} |\omega|^2 + |\nabla \omega|^2.
\]

Bootstrapping (formally):

\[
\mathcal{E} = ||\omega||_{L^2}^2 + ||a||_{L^2}^2 + ||\partial_t \omega||_{L^2}^2 + ||\partial_t a||_{L^2}^2
\]
\[
\mathcal{D} = ||\omega||_{H^1}^2 + ||\partial_t \omega||_{H^1}^2 \gtrsim ||a||_{H^{-1}}^2 + ||\partial_t a||_{H^1}^2
\]
Stable case – Linear analysis: hypo-coercivity

Recall:

\[ E = ||\omega||_{L^2}^2 + ||a||_{L^2}^2 + ||\partial_t \omega||_{L^2}^2 + ||\partial_t a||_{L^2}^2 \]
\[ D = ||\omega||_{H^1}^2 + ||\partial_t \omega||_{H^1}^2 \gtrsim ||a||_{H^{-1}}^2 + ||\partial_t a||_{H^1}^2 \]

Hypo-coercivity

By interpolation (formally):

\[ ||a||_{L^2}^2 \lesssim ||a||_{H^{-1}}^{2\theta} ||a||_{H^s}^{2(1-\theta)} \]

where \( \theta = \frac{s}{1+s} \uparrow 1 \) as \( s \uparrow \infty \). Therefore

\[ E \lesssim D^\theta E_{\text{high}}^{1-\theta} \Rightarrow E(t) \lesssim \frac{E(0)}{(1+t)^\alpha}, \quad \alpha = \frac{\theta}{1-\theta} \uparrow \infty \text{ as } \theta \uparrow 1. \]
Stable case – Energy estimates

Recall: the equation for $\partial_t \omega$ is

$$J(\partial_t + u \cdot \nabla)\omega + \omega \times J\omega = \tau e_3 + \kappa \nabla \times u - 2\kappa \omega$$

$$+ (\tilde{\alpha} - \tilde{\gamma}) \nabla (\nabla \cdot \omega) + \tilde{\gamma} \Delta \omega.$$ 

Question: for $\theta \sim \partial^\alpha \omega$, which terms do we keep?

A: Because of the precession, we keep an “advection-rotation” operator. If $\theta$ solves

$$J(\partial_t + u \cdot \nabla)\theta + (\omega \times J)\theta = C$$

subject to $\nabla \cdot u = 0$ and $(\partial_t + u \cdot \nabla)J = [\Omega, J]$, where $\omega = \text{vec} \Omega$, then

$$\frac{d}{dt} \int_{T^3} \frac{1}{2} J \theta \cdot \theta = \int_{T^3} C \cdot \theta.$$
Stable case – Energy estimates (continued)

For the full system involving \((u, \omega, J)\) and including dissipation, the energy estimate we obtain is (neglecting commutators),

\[
\frac{d}{dt} \int_{T^3} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} J \omega \cdot \omega + \frac{C}{(\nu - \lambda)} |a|^2 \right)
= -\int_{T^3} \left( \frac{\mu}{2} |\mathbb{D}u|^2 + 2\kappa \left( \frac{1}{2} \nabla \times u - \omega \right)^2 + \alpha |\nabla \cdot \omega|^2 + \frac{\beta}{2} |\mathbb{D}^0 \omega|^2 + 2\gamma |\nabla \times \omega|^2 \right).
\]

where \(a = (J_{12}, J_{23})\).

This gives us control over \(H^k\) norms of \((u, \omega, a)\).
Stable case – Transport estimates

If $u$ is divergence-free and $J$ solves the advection-rotation equation

$$(\partial_t + u \cdot \nabla - [\Omega, \cdot])J = M \text{ on } \mathbb{T}^3,$$

where $M$ is a symmetric matrix field, then

$$\frac{d}{dt} \|J\|_{L^p(\mathbb{T}^3)} \leq \|M\|_{L^p(\mathbb{T}^3)}.$$

Key: $[A, S] : S = 0$ for any skew-symmetric $A$ and symmetric $S$.

Combined with high-low estimates, this gives us control over $H^k$ norms of $J$ (in a small energy regime):

$$\|J(t)\|_{H^k} \lesssim \|J_0\|_{H^k} + \int_0^t \|(u, \omega)(s)\|_{H^k} ds.$$
Stable case – Controlling the interactions

Combining the energy and transport estimates and taking into account the hypo-coercivity the following picture emerges.

<table>
<thead>
<tr>
<th>low regularity</th>
<th>high regularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(u, \omega, a)$</td>
<td>decaying</td>
</tr>
<tr>
<td>$J$</td>
<td>bounded</td>
</tr>
</tbody>
</table>

Interactions – goal: $|I| \lesssim \sqrt{\mathcal{E}D}$ (because $\mathcal{E} + D = I$).

$$I = \int_{\mathbb{T}^3} (\partial_x^\alpha J) \omega \cdot \partial_x^\alpha a \sim \int_{\mathbb{T}^3} (\partial_x^{\alpha+1} J) \omega \cdot \partial_x^{\alpha-1} a$$

Only holds in a time-integrated fashion, i.e. $\int_0^t |I| \lesssim \mathcal{E}_0 \int_0^t D$. 
Stable case – Result (Tice and R.T., 2020)

Suppose that the micro-structure is pancake-like, i.e. $\nu > \lambda$, and that

$$X_{eq} = (u_{eq}, \omega_{eq}, a_{eq}) = \left(0, \frac{T}{2\kappa}e_3, \text{diag}(\lambda, \lambda, \nu)\right)$$

is the equilibrium solution. For any $k \geq 8$ there exists $\delta > 0$ such that the equations are globally well-posed in a $\delta$-ball in $H^k$ about $X_{eq}$. Moreover $(u, \omega, a)$ approach equilibrium at a rate $(1 + t)^{k-s}$ in $H^s$ (for $0 \leq s < k$).

The role of anisotropy

Anisotropy leads to

- hypo-coercivity at the linear level,
- estimates involving advection-rotation operators, and
- hyperbolic-parabolic interaction at the nonlinear level.
Conclusion
Conclusion

- Rigid microstructure + continuum mechanics = micropolar fluids.
- Subject to a uniform micro-torque, we obtain a sharp nonlinear stability criterion for axially symmetric microstructure.

##### Anisotropy

Fear leads to the dark side.

– Yoda

Rod-like: unstable  Pancake-like: stable

- **Anisotropy** leads to several difficulties:

<table>
<thead>
<tr>
<th></th>
<th>Unstable case</th>
<th>Stable case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear level</td>
<td>not self-adjoint</td>
<td>hypo-coercivity</td>
</tr>
<tr>
<td>Nonlinear level</td>
<td>no spectral gap</td>
<td>parabolic-hyperbolic interactions</td>
</tr>
</tbody>
</table>

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Thank you for your attention!