CHAPTER 1

A short treatise on micropolar continuum mechanics

ABSTRACT.
We derive the equations of motion of Newtonian incompressible homogeneous
micropolar fluids by following the path of rational continuum mechanics. By
contrast with classical fluids, micropolar fluids allows for the non-trivial behaviour
of a rigid microstructure at the microscopic scale. This introduces an additional
kinematic quantity, an additional conserved quantity, and an additional stress
tensor responsible for the mediation of couples at the microscopic scale, namely
the angular velocity and the microinertia of the microstructure and the couple
stress tensor, respectively.
To be more precise, we derive the equations by postulating (1) the integral balance
laws for conserved physical quantities such as mass, linear, and angular momen-
tum, (2) the frame-invariance of the constitutive equations for the stress and
couple-stress tensor, and (3) the satisfaction of the Onsager reciprocity relations.
Micropolar fluids were introduced by Eringen in [Eri66] as part of an effort to describe microcontinuum mechanics, which extend classical continuum mechanics by taking into account the effects of microstructure present in the medium. For viscous, incompressible continua, this results in a model in which the incompressible Navier-Stokes equations are coupled to an evolution equation for the rigid microstructure present at every point of the continuum. This theory can be used to describe aerosols and colloidal suspensions such as those appearing in biological fluids [Mau85], blood flow [Ram85, BBR+08, MK08], lubrication [AK71, BL96, NS12] and in particular the lubrication of human joints [SSP82], liquid crystals [Eri66, LR04, GBRT13], and ferromagnetic fluids [NST16].

In this chapter we carefully derive the equations governing micropolar fluids in the spirit of rational continuum mechanics, under the additional assumption that the fluid is incompressible and homogeneous.

Disclaimer. It is important to note that the derivation of the equations of motion for micropolar fluids is heavily inspired by the sections of [Eri99, Eri01] relevant to micropolar fluids. To a large degree, many portions of this chapter are more mathematically-minded reformulations of Eringen’s original description of micropolar fluids.

1. Introduction

In this introduction we briefly sketch the derivation of the equations of motion of micropolar fluids, pointing out the relevant sections of this chapter where more details can be found regarding each step in the derivation. We will also highlight the differences between micropolar continua and classical continua throughout. This introduction will contain numerous links to later portion of this chapter, pointing to the precise definition of the notions discussed here.

The story of micropolar continuum mechanics, like that of classical continuum mechanics, begins with a kinematic description of the continuum. In the classical realm, this means that a continuum is fully determined by a flow map \( \eta : [0, \infty) \to \mathbb{R}^n \to \mathbb{R}^n \) which determines the motion of the continuum. In the micropolar realm this description is supplemented by a microrotation map \( Q : [0, \infty) \times \mathbb{R}^n \to SO(n) \) which assigns an orientation to every point in the micropolar continuum. This is illustrated in Figure 1. This kinematic description of continua and micropolar continua is detailed in Section 2, which also contains an elementary discussion of rigid motions and incompressible flows, two fundamental classes of continua. Once a kinematic description of micropolar continua is established we throw physics into the mix. The objective,
2. KINEMATICS

at this stage, is to postulate appropriate conservation laws for various physical quantities associated with the continuum. In order to do this we must first (1) carefully define these physical quantities and (2) carefully study them in the special case where the (classical) continuum is a rigid body. This is essential since the defining feature of micropolar continua is the presence of rigid microstructure. A good understanding of the physics of rigid bodies is therefore essential to ensure that the conservation laws which we will posit to hold for micropolar continua are physically sound. We define these physical quantities, namely mass, moment of inertia, linear momentum, and angular momentum in Section 3.

With these tools in hand we may now postulate appropriate conservation laws for micropolar continua. These conservation laws are formulated as integral balance laws and so the key point at the stage is to derive the corresponding local differential equations satisfied by the micropolar continuum. The formulation of the balance laws and the derivation of their local counterparts is carried out in Section 4. In particular, while classical continua are taken to conserve mass, linear momentum, and angular momentum, micropolar continua are posited to also conserve microinertia, which is defined to be the moment of inertia of the microstructure.

Note that throughout this treatment of micropolar fluids we deliberately avoid discussing boundary conditions. We make an exception in Section 4 to derive the natural boundary conditions that naturally arise from the conservation laws. This is done solely so that we may ultimately obtain a complete set of partial differential equations, and we do not focus on boundary effects in this chapter.

At this stage we already have in hand the complete set of equations satisfied by the micropolar continuum. However the balances of linear and angular momentum introduced two additional unknowns: the stress tensor \( T \) and the couple stress tensor \( M \).

Physically, these tensors encode the response of the micropolar fluid to the forces and the torques induced by the neighbouring fluid. Note that only the stress tensor appears in classical fluids since in that case there is no microstructure that can support torques and hence no way for the fluid to apply torques to itself. Mathematically, these tensors render the system overdetermined, pending constitutive relations which determine \( T \) and \( M \) in terms of the dynamic variables. In the micropolar world there are two dynamic variables at play: the velocity \( u \) and angular velocity \( \omega \), which are essentially time-derivatives of the flow map and microrotation map, respectively.

To conclude we therefore impose three constraints on the stress tensor \( T \) and \( M \): (1) they only depend on the dynamic variables and their gradients, and in a linear fashion (this is the Newtonian assumption), (2) their dependence on the dynamic variables is frame-invariant, i.e. independent of the frame of reference used to observe the fluid, and (3) they respect the Onsager reciprocity relations (which is a thermodynamical restriction). The constraints (1) and (2) are familiar from classical Newtonian fluids but (3) may not be, so we direct the reader’s attention to Section 5.3 for a more detailed discussion of the Onsager reciprocity relations. The determination of the necessary forms of \( T \) and \( M \) given the constraints (1)–(3) is carried out in Section 5. We conclude Section 5 by recording the equations of motion of micropolar fluids in Corollary 5.21.

The last section of Chapter 1, namely Section 6, is a short appendix collecting various identities which are either well-known or elementary and which are used elsewhere in Chapter 1.

2. Kinematics

In this section we introduce the fundamental objects used to describe continua and micropolar continua, in Section 2.1 and Section 2.3 respectively. We also take the time to carefully discuss two important classes of continua, namely rigid motions and incompressible flows in Section 2.2.

2.1. Continua. We begin with a discussion of continua, which are defined below in Definition 2.1. In this section we also record elementary results regarding derivatives of functions defined along the flow of a continuum.

DEFINITION 2.1. (Continuum)

A **continuum** is a pair \( (\Omega_0, \eta) \) where:
(1) \( \Omega_0 \) is open, with \( \partial \Omega_0 \) Lipschitz, and is called a reference configuration.
(2) \( \eta : [0, \infty) \times \Omega_0 \to \mathbb{R}^n \) is a map such that for every \( t \geq 0 \), \( \eta_t := \eta(t, \cdot) \) is an orientation-preserving \( C^1 \)-diffeomorphism onto its image, called a flow map.

We write \( \Omega(t) := \eta_t(\Omega) \).
With the definition of a continuum in hand we can define what it means for functions and measures to be defined “along the flow”. Put simply, and informally, a function is defined along the flow of a continuum \((\Omega, \eta)\) if its domain is \([0, \infty) \times \Omega(t)\). The precise definition is below in Definition 2.2.

**Definition 2.2.** (Functions and measures defined ‘along the flow’)
Let \((\Omega_0, \eta)\) be a continuum.
1. A map \(f : [0, \infty) \times \Omega_0 \to X\) is a called a Lagrangian function defined along the flow.
2. A collection of measures \(\mu = (\mu_t)_{t \geq 0}\) such that for every \(t \geq 0\), \(\mu_t\) is a measure on \(\Omega_0\) is called a Lagrangian measure defined along the flow.
3. We say that \(g\) is an Eulerian function defined along the flow if there exists a Lagrangian function \(f\) defined along the flow such that \(g(t, x) = f(t, \eta_t^{-1}(x))\), i.e. for every \(t \geq 0\), \(g(t, \cdot)\) is a map on \(\Omega(t)\). We summarize this informally by writing \(g = f \circ \eta_t^{-1}\).
4. We say that \(\nu\) is an Eulerian measure defined along the flow if there exists a Lagrangian measure \(\mu\) defined along the flow such that \(\nu_t = (\eta_t)^\# \mu_t\), i.e. for every \(t \geq 0\), \(\nu_t\) is a measure on \(\Omega(t)\). We summarize this informally by writing \(\nu = \eta_t^\# \mu\).

We now introduce the notion of Lagrangian and Eulerian coordinates. The motivation behind Lagrangian and Eulerian coordinates is that the former correspond to coordinates in the continuum’s reference configuration \(\Omega_0\) whilst the latter correspond to coordinates in an observer’s frame of reference where the complete history of the trajectory of each point of the continuum is not kept track of.

**Definition 2.3.** (Lagrangian and Eulerian coordinates)
Let \((\Omega_0, \eta)\) be a continuum.
1. \((t, y) \in [0, \infty) \times \Omega_0\) are called Lagrangian coordinates.
2. \((t, x) \in [0, \infty) \times \Omega(t)\) are called Eulerian coordinates.

We continue this initial avalanche of definitions by introducing the velocity and acceleration of a continuum. These notions are absolutely fundamental for our purposes here: we seek equations of motion for micropolar fluids and one of these unknowns will be precisely the Eulerian velocity.

**Definition 2.4.** (Velocity and acceleration)
Let \((\Omega_0, \eta)\) be a continuum.
1. \(v := \partial_\eta : [0, \infty) \times \Omega_0 \to \mathbb{R}^3\) is called the Lagrangian velocity.
2. \(a := \partial_\eta v = \partial^2_\eta : [0, \infty) \times \Omega_0 \to \mathbb{R}^3\) is called the Lagrangian acceleration.
3. \(u : [0, \infty) \times \Omega(t)\), defined via, for every \(t \geq 0\), \(u_t := \eta_t \circ \eta_t^{-1}\), is called the Eulerian velocity.
4. \(b : [0, \infty) \times \Omega(t)\), defined via, for every \(t \geq 0\), \(b_t := a_t \circ \eta_t^{-1}\), is called the Eulerian acceleration.

In Lagrangian coordinates the acceleration \(a\) and the velocity \(v\) satisfy the familiar relation \(a = \partial_t v\). This picture is slightly more complicated in Eulerian coordinates since the relationship between the Eulerian acceleration \(b\) and the Eulerian velocity \(u\) is given by \(b = \partial_t u + (u \cdot \nabla) u\). Indeed we can verify from Proposition 2.6 below that
\[
a = \partial_t v = \left((\partial_t + u \cdot \nabla) u\right) \circ \eta
\]
and hence \(b = a \circ \eta^{-1} = (\partial_t + u \cdot \nabla) u\). This means that the Eulerian acceleration is the material derivative of the Eulerian velocity. Actually, the operators \(f \mapsto \partial_t f + (u \cdot \nabla) f\) and its closely related cousin \(f \mapsto \partial_t f + \nabla \cdot (fu)\), which we refer to as material derivatives, occur so often in continuum mechanics that they are given their own notation - see Definition 2.5 below.

**Definition 2.5.** (Material derivatives)
Let \(T\) be a tensor field differentiable in both space and time and let \(u\) be a vector field. We define
\[
D^\mu T := \partial_\mu T + (u \cdot \nabla) T \quad \text{and} \quad D^\mu := \partial_\mu + \nabla \cdot (T \otimes u)
\]
where we use the notation \((T \otimes u)_{i_1 \ldots i_n} := T_{i_1 \ldots i_n} u_j\).

Now that we have introduce the fundamental objects of continuum mechanics we start recording some fundamental results associated with their derivatives. In particular in Proposition 2.6 below we record an identity for the temporal derivatives of functions and measures defined along the flow.

**Proposition 2.6.** (Derivatives along a flow)
Let \((\Omega_0, \eta)\) be a continuum with Eulerian velocity field \(u\).
(1) **(Derivatives of functions defined along the flow)**

For any \( f : [0, \infty) \times \Omega(t) \to \mathbb{R} \),

\[
\frac{d}{dt} (f \circ \eta) = \left( (\partial_t + u \cdot \nabla) f \right) \circ \eta.
\]

(2) **(Derivatives of the volume form, i.e. of the Lebesgue measure, along the flow)**

For any Lebesgue-measurable \( E_0 \subseteq \Omega_0 \), writing \( E(t) := \eta_t(E_0) \), we have that

\[
\frac{d}{dt} \mathcal{L}^n(E(t)) = \left( (\nabla \cdot u) d\mathcal{L}^n \right)(E(t))
\]

where the measure denoted \( fd\mu \) is defined via \( (fd\mu)(A) := \int_A f d\mu \).

Note that this boils down to the following equation, which is essentially a pointwise version of the equation above:

\[
\frac{d}{dt} \det \nabla \eta = \left( (\nabla \cdot u) \circ \eta \right) \det \nabla \eta.
\]

**Proof.** (1) follows from an immediate computation upon recalling that \( \partial_t \eta = u \circ \eta \):

\[
\frac{d}{dt} f \circ \eta = \frac{d}{dt} f(\eta(y,t),t) = (\nabla f \circ \eta) \cdot \partial_t \eta + \partial_t f \circ \eta = ((\partial_t + u \cdot \nabla) f) \circ \eta.
\]

To derive (2), recall from Corollary 6.22 that \( \det|_{M_0}(A) = \det(M_0) \text{tr}(M_0^{-1} A) \). Therefore

\[
\frac{d}{dt} \det \nabla \eta = \det|_{\nabla \eta}(\partial_t \nabla \eta) = \det (\nabla \eta) \text{tr} \left( (\nabla \eta)^{-1} \partial_t \nabla \eta \right)
\]

where

\[
\partial_t \nabla \eta \cdot (\nabla \eta)^{-1} = \nabla \partial_t \eta \cdot (\nabla (\eta^{-1}) \circ \eta) = \left( (\nabla \partial_t \eta \circ \eta^{-1}) \cdot \nabla (\eta^{-1}) \right) \circ \eta = (\nabla (\partial_t \eta \circ \eta^{-1})) \circ \eta = \nabla u \circ \eta
\]

and hence

\[
\frac{d}{dt} \det \nabla \eta = \det (\nabla \eta) \text{tr}(\nabla u \circ \eta) = (\nabla \cdot u) \circ \eta \det \nabla \eta.
\]

So finally

\[
\frac{d}{dt} \mathcal{L}^n(E(t)) = \frac{d}{dt} \int_{E(t)} d\mathcal{L}^n = \frac{d}{dt} \int_{E_0} \det(\nabla \eta) d\mathcal{L}^n = \int_{E_0} (\nabla \cdot u) \circ \eta \det(\nabla \eta) d\mathcal{L}^n = \int_{E(t)} (\nabla \cdot u) d\mathcal{L}^n = ((\nabla \cdot u) d\mathcal{L}^n)(E(t))
\]

\( \square \)

Using Proposition 2.6 above we are now equipped to prove a result which will be absolutely essential when it comes to deriving local versions of the integral balance laws. Indeed, Theorem 2.7 below tells us precisely how to “push time derivatives inside integrals defined over domains carried by the flow”.

**Theorem 2.7.** *(Reynolds’ transport theorem)*

Let \( \mathcal{U}_0 \subseteq \mathbb{R}^n \) be open and let \( \eta : [0, \infty) \times \mathcal{U}_0 \to \mathbb{R}^n \) be a map such that, for every \( t \geq 0 \), \( \eta_t := \eta(t, \cdot) \) is an orientation-preserving \( C^1 \)-diffeomorphism. Define \( \mathcal{U}(t) := \eta_t(\mathcal{U}_0) \), and \( u_t := \partial_t \eta_t \circ \eta_t^{-1} \) for every \( t \geq 0 \). For any sufficiently regular \( f : [0, \infty) \times \mathcal{U}(t) \to \mathbb{R} \),

\[
\frac{d}{dt} \int_{\mathcal{U}(t)} f = \int_{\mathcal{U}(t)} \partial_t f + \nabla \cdot (fu).
\]

**Proof.** Equipped with Proposition 2.6, this is a direct computation

\[
\frac{d}{dt} \int_{\mathcal{U}(t)} f = \frac{d}{dt} \int_{\mathcal{U}_0} (f \circ \eta) \det \nabla \eta = \int_{\mathcal{U}_0} ((\partial_t + u \cdot \nabla) f \circ \eta) \det \nabla \eta + f(\nabla \cdot u) \circ \eta \det \nabla \eta
\]

\[
= \int_{\mathcal{U}(t)} \partial_t f + \nabla(f) \cdot u + f(\nabla \cdot u) = \int_{\mathcal{U}(t)} \partial_t f + \nabla \cdot (fu).
\]

\( \square \)

We can interpret Proposition 2.6 and Theorem 2.7 above as telling us how to differentiate 0-forms and \( n \)-forms, i.e. smooth functions and volume forms respectively, in terms of the material derivatives introduced in Definition 2.5. More precisely: for a 0-form \( f \) and its associated \( n \)-form \( fd\mathcal{L}^n \), Proposition 2.6 and Theorem 2.7 tell us that, if \( \mathcal{U}(t) \) denotes a subset of \( \Omega(t) \) carried by the flow (i.e. \( \mathcal{U}(t) = \eta_t(\mathcal{U}_0) \) for some \( \mathcal{U}_0 \subseteq \Omega_0 \)) then

\[
\partial_t(f \circ \eta) = (D^n f \circ \eta) \text{ and } \partial_t \left( fd\mathcal{L}^n|_{\mathcal{U}(t)} \right) = (D^n_t f) d\mathcal{L}^n|_{\mathcal{U}(t)}.
\]
In particular, note that when $\nabla \cdot u$ vanishes (i.e. the flows is incompressible – see Definition 2.12 below) then the two material derivatives agree, i.e. $D^t_1 = D^t_0$, and therefore 0-forms and $n$-forms behave in the same way when differentiated in time along the flow. This is a manifestation of a feature of incompressible flows that we will prove below (in Proposition 2.15), namely that incompressible flows preserve the Lebesgue measure.

2.2. Important classes of continua: rigid motions and incompressible flows. In this section we introduce rigid motions and incompressible flows, which are two fundamental examples of continua. We also establish some of their properties, and in particular will establish the implications laid out in the diagram below:

![Diagram: Rigid motion, Isometric flow, Locally volume-preserving flow, u ∈ ker D, \nabla \cdot u = 0 (incompressible flow)]

Rigid motions and rigid bodies, introduced in Definition 2.8 below, are important for several reasons.

1. They are the simplest class of continua that one can study.
2. They motivate and help to illustrate numerous definitions in the realm of micropolar continua, such as the definitions of linear momentum and angular momentum. Indeed, the microstructure of micropolar fluids is posited to be a microscopic rigid body, so a good understanding of rigid bodies is important in motivating the defining properties of micropolar media.
3. They are necessary to define the notion of frame-invariance, which plays in essential role in the determination of the equations of motion of micropolar fluids.

**Definition 2.8.** (Rigid motion and rigid body)

1. We say that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rigid motion if $f = R + z$ for some $R \in O(n)$ and $z \in \mathbb{R}^n$. Note that $f$ is orientation-preserving if and only if $R \in SO(n)$.
2. We say that a flow map $\eta : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rigid motion if $\eta_t$ is a rigid motion for every $t \geq 0$.
3. A continuum whose flow map is a rigid motion is called a rigid body.

The first property of rigid motions we establish is that rigid motions are precisely the isometries of $\mathbb{R}^n$. This also serves as a justification for the central important of rigid motions.

**Proposition 2.9.** Rigid motions are precisely the isometries of Euclidean space, and moreover isometries of Euclidean space are necessarily bijective.

**Proof.** First we show that rigid motions are isometries. Let $f = R + z$ be a rigid motion. Then, for every $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| = |Rx - Ry| = |x - y|$ since $R$ is orthogonal and hence

$$|R(x - y)| = (R(x - y) \cdot R(x - y))^{1/2} = (R^T R (x - y) \cdot (x - y))^{1/2} = |x - y|$$

such that indeed $f$ is an isometry.

Now we show that isometries must rigid motions. We proceed in several steps. In step 1 we use a polarization identity to show that isometries preserve inner products (i.e. angles). In step 2 we deduce that isometries must therefore be additive. In step 3 we note that isometries must be continuous and injective. In step 4 we deduce from the additivity and the continuity of the isometry that it must be linear. We finally conclude in step 5.

**Step 1.** Let $f$ be an isometry of $\mathbb{R}^n$. We will use a polarization identity, which allows us to relate inner products and norms, to show that $f$ also preserves the inner product. Suppose without loss of generality that $f(0) = 0$. Then, for any $x, y \in \mathbb{R}^n$,

$$x \cdot y = \frac{1}{2} (|x|^2 + |y|^2 - |x - y|^2) = \frac{1}{2} \left( |x - 0|^2 + |y - 0|^2 - |x - y|^2 \right)$$

$$= \frac{1}{2} \left( (f(x) - f(0))^2 + (f(y) - f(0))^2 - |f(x) - f(y)|^2 \right)$$

$$= \frac{1}{2} \left( |f(x)|^2 + |f(y)|^2 - |f(x) - f(y)|^2 \right) = f(x) \cdot f(y).$$
Step 2. We now show that $f$ is additive. For any $x, y, z \in \mathbb{R}^n$,
\[
|(x + y) - z|^2 = |x|^2 + 2x \cdot y + |y|^2 - 2(x \cdot z + y \cdot z) + |z|^2
\]
\[
= |f(x)|^2 + 2f(x) \cdot f(y) + |f(y)|^2 - 2(f(x) \cdot f(z) + f(y) \cdot f(z)) + |f(z)|^2
\]
and thus in particular if we pick $z = x + y$ we obtain that
\[
f(x) + f(y) - f(x + y) = 0
\]

Step 3. We now note that, since $f$ is an isometry, it must be a continuous injection. For every $x \neq y$, $x, y \in \mathbb{R}^n$ \(|f(x) - f(y)| = |x - y| > 0$$ and hence $$f(x) \neq f(y)$$. Continuity is immediate since isometries are 1-Lipschitz.

Step 4. We now show that, since $f$ is additive and continuous, it must be 1-homogeneous, and hence linear. Let $f$ be additive and continuous. Then, for any $x \in \mathbb{R}^n$, and for any $z \in \mathbb{Z}$, $f(\frac{X}{z}) = zf(x)$, and hence $zf(\frac{X}{z}) = f(\frac{X}{z^2}) = f(x)$. It follows that $f(\frac{X}{z^2}) = \frac{1}{z^2}f(x)$, and thus $f(qx) = qf(x)$ for any rational $q$. Finally, by continuity of $f$ and density of $\mathbb{Q}$ in $\mathbb{R}$, we obtain that $f(\lambda x) = \lambda f(x)$ for any $\lambda \in \mathbb{R}$. So indeed $f$ is 1-homogeneous. Since it is also additive, it must be linear.

Step 5. We may now conclude the argument and show that $f$ is a rigid motion. We know that $f$ is linear and injective. It is therefore bijective (since the dimensions of the domain and codomain of $f$ coincide). So let us write $f(x) = Ax$ for some $n$-by-$n$ matrix $A$. We note that, for arbitrary $x, y \in \mathbb{R}^n$,
\[
x \cdot y = f(x) \cdot f(y) = Ax \cdot Ay = A^T Ax \cdot y
\]
and hence $A$ is indeed an orthogonal matrix. \qed

We now introduce the symmetrized gradient. Due to its connections with rigid motions with frame-invariance, this differential operator appears throughout our treatment of micropolar continuum mechanics.

**Definition 2.10.** (Symmetrized gradient)
For any differentiable vector field $v$ we define its symmetrized gradient, denoted $\mathbb{D}v$, to be $(\mathbb{D}v)_{ij} = \partial_i v_j + \partial_j v_i$, i.e. $\mathbb{D}v = 2 \text{Sym} \nabla v$.

Having defined the symmetrized gradient we show that it can be used to characterize the Eulerian velocities of rigid motions.

**Proposition 2.11.** A flow map is a rigid motion if and only if its Eulerian velocity belongs to the kernel of the symmetrized gradient.

**Proof.** ($\Rightarrow$) Suppose $\eta(t, y) = z(t) + R(t)y$ for some $z : [0, \infty) \to \mathbb{R}^n$ and some $R : [0, \infty) \to O(n)$. Note that $\eta^{-1}(t, x) = R(t)(x - z(t))$, and therefore:
\[
u(t, x) = (\partial_\eta \circ \eta^{-1})(t, x) = \dot{z}(t) + \dot{R}(t)R(t)^{-1}(x - z(t))
\]
i.e. $u(t, \cdot) = v + \Omega$ for
\[
\begin{align*}
v &= \dot{z} - \dot{R}R^{-1}z \\
\Omega &= \dot{R}R^{-1}
\end{align*}
\]
(2.1)

Now note that, by Proposition 6.18, $\dot{R} = RB$ for some $B \in O(n)$, and hence by Proposition 6.19 $\Omega = RBR^{-1} \in A(n)$. So finally, by Lemma 6.6, $u \in \ker \mathbb{D}$.

($\Leftarrow$) This direction of the proof amounts to solving the ODE (2.1), treating $(\Omega, v)$ as data and $(R, z)$ as unknowns, with initial conditions $z(0) = 0$ and $R(0) = I$ (to ensure that $\eta_0 = \text{id}$). So let $u \in \ker \mathbb{D}$, i.e by Lemma 6.6, $u(t, x) = v(t) + \Omega(t)x$ for some $v : [0, \infty) \to \mathbb{R}^n$ and some $\Omega : [0, \infty) \to \text{Skew} (\mathbb{R}^{n \times n})$. Then define, for $t \geq 0$,
\[
z(t) = \int_0^t e^{\Omega(t-s)}v(s)\,ds
\]
i.e. $z(t) = R(t)\int_0^t R(s)^{-1}v(s)\,ds$. Upon taking a derivative we see that
\[
\dot{z} = RR^{-1}v + \dot{R}R^{-1}z
\]
such that (2.1) holds indeed. \qed
Having discussed rigid motions we now introduce the second of the two classes of continua that we consider in this section, namely incompressible flows.

**Definition 2.12. (Incompressible flow)**
We say that a flow map $\eta$ is incompressible if its Eulerian velocity is divergence-free.

We now define what it means for a flow to be locally volume-preserving, a notion which will play with incompressible flows the role that isometries played with rigid motions.

**Definition 2.13. (Locally volume-preserving maps and flows)**
1. Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable. A map $f : E \to \mathbb{R}^n$ is said to be locally volume-preserving if $f_\#\mathcal{L}^n = \mathcal{L}^n$.
2. A flow map $\eta$ is said to be locally volume-preserving if, for every $t \geq 0$, $\eta_t$ is locally volume-preserving, i.e. $(\eta_t)_\#\mathcal{L}^n = \mathcal{L}^n$.

Having introduced locally volume-preserving flows we record a few different equivalent characterizations of such flows. This will be helpful later when discussing the relationships between locally volume-preserving flows, incompressible flows, and isometries.

**Lemma 2.14. (Alternative characterizations of locally volume-preserving flow maps)**
Let $\eta$ be a flow map. The following are equivalent:
1. $\eta$ is locally volume-preserving.
2. For every Lebesgue-measurable set $E_0 \subseteq \Omega_0$, writing $E (t) := \eta_t (E_0)$, we have that
   \[ \mathcal{L}^n (E_0) = \mathcal{L}^n (E (t)). \]
3. $|\det \nabla \eta| \equiv 1$.

**Proof.** (1) $\iff$ (2) This follows from the observation that, since $\eta_t$ is a bijection for every $t \geq 0$,
   \[ \mathcal{L}^n (E_0) = \mathcal{L}^n (\eta_t^{-1} (E (t))) = (\eta_t)_\# \mathcal{L}^n (E (t)). \]

(2) $\iff$ (3) This follows from the observation that
   \[ \mathcal{L}^n (E (t)) = \int_{E(t)} d\mathcal{L}^n = \int_{E_0} |\det \nabla \eta| d\mathcal{L}^n = (|\det \nabla \eta| d\mathcal{L}^n) (E_0). \]

We now show an analog of Proposition 2.9 in the realm of incompressible flows.

**Proposition 2.15.** A flow map is locally volume-preserving if and only if it is incompressible.

**Proof.** This follows from combining one of the alternative characterization of locally volume-preserving flows in Lemma 2.14 which says that a flow map $\eta$ is locally volume-preserving if and only if $|\det \nabla \eta| \equiv 1$, with the computation of the time derivative of the volume form in Proposition 2.6, which tells us that
\[ \frac{d}{dt} \det \nabla \eta = (\nabla \cdot u) \omega \det \nabla \eta. \] Note that we also need to use the fact that $\eta_t = \text{id}$, and hence $\det \nabla \eta_0 \equiv 1$. 

Finally we conclude this section by remarking on the relationship between locally volume-preserving isometric flow maps.

**Proposition 2.16.** Isometric flow maps are locally volume-preserving.

**Proof.** Since $\eta$ is an isometry, it follows from Proposition 2.9 that it is a rigid motion, i.e.
\[ \eta (t, x) = z (t) + R (t) x \]
for some $z : [0, \infty) \to \mathbb{R}^n$ and some $R : [0, \infty) \to O (n)$. In particular, $|\det \nabla \eta| = |\det R| = |\pm 1| = 1$, which by Lemma 2.14 tells us precisely that $\eta$ is locally volume-preserving.
2. Kinematics

2.3. Micropolar Continua. In this section we follow a path similar to that which we took in Section 2.1 earlier: we define the fundamental kinematic objects used to describe micropolar continua. This begins with the definition of a micropolar continuum.

**Definition 2.17. (Micropolar continuum and microrotation map)**

A **micropolar continuum** is a triple \((\Omega_0, \eta, Q)\) where

1. \((\Omega_0, \eta)\) is a continuum.
2. \(Q : \Omega_0 \times [0, \infty) \to SO(n)\) is called a **microrotation map**.

A word of warning: there are two ways to define the microrotation map and we have chosen here the convention that \(Q\) is **absolute**. Indeed, one may either define \(Q\) to be the rotation of the microstructure with respect to its immediate environment, in which case \(Q\) would be equal to the identity when the micropolar continuum undergoes rigid motions such as rotations, or one may define \(Q\) to be the identity at time \(t = 0\) and to be the absolute rotation undergone by the micropolar continuum thereafter. We choose the latter convention. In order to illustrate the physical interpretation of the microrotation map \(Q\), Table 1 contrasts the motions obtained for various simple expressions of \(\eta\) and \(Q\).

We now introduce two linear maps, **ten** and **vec**, which will play a fundamental role throughout.

**Definition 2.18. (ten and vec)**

We define \(\text{ten} : \mathbb{R}^3 \to \text{Skew}(3)\) and \(\text{vec} : \text{Skew}(3) \to \mathbb{R}^3\) via: for every \(v \in \mathbb{R}^3\) and every \(A \in \text{Skew}(3)\),

\[
(\text{ten}v)_{ij} := \epsilon_{iaj}v_a \quad \text{and} \quad (\text{vec}A)_i := \frac{1}{2}\epsilon_{aib}A_{ab}.
\]

The linear maps **ten** and **vec** are essential since they allow us, in light of Proposition 6.13, to identify \(\mathbb{R}^3\) with **Skew(3)**, the space of 3-by-3 skew-symmetric matrices. Quantities like angular velocity and angular momentum, that would naturally take the form of a skew-symmetric matrix (since they arise from the rotational invariance of physical system and, as noted in Proposition 6.19, the tangent space to the space of orthogonal matrices is precisely the space of skew-symmetric matrices), will thus be treated as vectors.
We conclude this section with the analog of Definition 2.4 for the micropolar realm and introduce the
dynamic quantities that can be used to describe the motion of the microstructure of micropolar continua.

**Definition 2.19.** (Angular velocity and angular velocity tensor)
Let \( (\Omega_0, \eta, Q) \) be a micropolar continuum.
1. If \( \Theta := (\partial_t Q)^{-1} : \Omega_0 \times [0, \infty) \to A(n) \) is called the Lagrangian angular velocity tensor.
2. If \( n = 3, \theta := \text{vec} \Theta : \Omega_0 \times [0, \infty) \to \mathbb{R}^3 \) is called the Lagrangian angular velocity.
3. \( \Theta : \Omega(t) \to A(n) \), defined for, every \( t \geq 0 \), \( \Omega_t := \Theta_t \circ \eta_t^{-1} \), is called the Eulerian angular velocity.
4. If \( n = 3, \omega : \Omega(t) \to \mathbb{R}^3 \), defined for, every \( t \geq 0 \), \( \omega_t := \text{vec} \Theta_t = \theta_t \circ \eta_t^{-1} \), is called the Eulerian angular velocity.

We can motivate the definition of the angular velocity tensors as follows. If a rigid motion \( f \) maps maps
points \( y \) in the reference configuration to
\[
x(t,y) = f(t,y) = b(t) + R(t)(y - b_0)
\]
for \( b_0 \in \mathbb{R}^n, b : [0, \infty) \to \mathbb{R}^n \), and \( R : [0, \infty) \to O(n) \), then \( \partial_t f(t,y) = \dot{b} + \dot{R} y \). In particular, since we may
invert (2.2) to write \( y(t,x) = R^T (x-b) + b_0 \), we deduce that
\[
\partial_t f(t,y(t,x)) = \dot{b} + \dot{R} R^T (x - b).
\]
Expressing the time derivative of \( f \) in these coordinates is not merely a sleight of hand: those are precisely
the coordinates in which we can measure \( f \) if we are not keeping track of the original position of each point \( x \). Crucially: this expression motivates defining the angular velocity of the rigid motion as \( RR^T \), which is
akin to how we defined the angular velocity tensors in Definition 2.19 above.

3. Physics and rigid bodies

In this section we introduce various physical quantities associated with continua and micropolar continua,
such as mass and moment of inertia in Section 3.1 and linear momentum and angular momentum in Section
3.3. In Section 3.2 we take care to characterize the admissible moments of inertia, i.e. determining precisely
which positive symmetric matrices are the moment of inertia of some continuum.

**3.1. Mass and moments of inertia.** In this section we introduce several concepts and quantities
related to mass and moment of inertia. We then take care to compute the values of these quantities for some
simple example, and we record how these quantities behave under rigid motions and Cartesian products.
This will come in handy in Section 3.2 when we seek to characterize the admissible moments of inertia.

We being by defining the mass, moment of inertia, and other associated objects for a continuum. Recall
that, physically-speaking, the mass and moment of inertia play very similar role. Each of these quantities is
a phenomenological constant which encodes the inertia response of a body to exerted forces and torques.

**Definition 3.1.** (Mass, center of mass, and associated notions)
1. Given a Borel measure \( \nu \) on \( \mathbb{R}^n \) we define, for every Borel set \( E \),
   (a) its mass, denoted \( \mathcal{M}(E) \), via \( \mathcal{M}(E) := \nu(E) \),
   (b) its center of mass, denoted \( \overline{x} \), via
   \[
   \overline{x} := \frac{1}{\mathcal{M}(E)} \int_E x \, d\nu(x) = \frac{\int_E x \, d\nu(x)}{\nu(E)} = \mathbb{E}_\nu [x]
   \]
   (c) its covariance matrix, denoted \( V \), via
   \[
   V := \int_E (x - \overline{x}) \otimes (x - \overline{x}) \, d\nu(x) = \mathbb{E}_\nu [(x - \mathbb{E}_\nu [x]) \otimes (x - \mathbb{E}_\nu [x])] = \mathbb{V}_\nu [x]
   \]
   (d) if \( n = 3 \), its moment of inertia, denoted \( J \), via
   \[
   J := \int_E (|x - \overline{x}|^2 - (x - \overline{x}) \otimes (x - \overline{x})) \, d\nu(x) = \mathcal{M}(E) ((\text{tr} \, V) I - V)
   \]
2. We say that a measure is *finite* if the mass of its support is finite.
3. We say that a measure has *finite second moment* if the covariance matrix of its support is finite.
(4) A Lagrangian measure $\mu$ defined along the flow such that, for every $t \geq 0$, $\mu_t$ is a finite Borel measure with finite second moment is also called a Lagrangian mass measure.

(5) If a Lagrangian mass measure $\mu$ is absolutely continuous with respect to the Lebesgue measure, meaning that for every $t$, $\mu_t \ll \mathcal{L}^n$, then we call the Lagrangian function $\sigma$ defined along the flow via, for all $t \geq 0$, $\sigma_t := \frac{d\mu_t}{d\mathcal{L}^n}$, the Lagrangian mass density associated with $\mu$.

(6) An Eulerian measure defined along the flow such that, for every $t \geq 0$, $\nu_t$ is a finite Borel measure with finite second moment is also called an Eulerian mass measure.

(7) If an Eulerian mass measure $\nu$ is absolutely continuous with respect to the Lebesgue measure, then we call the Eulerian function $\rho$ defined along the flow via, for all $t \geq 0$, $\rho_t := \frac{d\nu_t}{d\mathcal{L}^n}$, the Eulerian mass density associated with $\nu$.

Since the center of mass and covariance matrix of a finite measure with finite second moment are nothing more than the expectation and covariance matrix of the probability measure obtained by normalizing the measure, we will often refer to these quantities as statistic functionals of the measure.

**Remark 3.2.** The fact that the moment of inertia $J$ has a form which is, at first sight, somewhat odd is worth remarking on. This particular form of $J$ is a consequence of our insistence to identify $\text{Skew}(3)$ with $\mathbb{R}^3$. Indeed: the natural space for the angular velocity tensor $\Omega$ to live in is $\text{Skew}(3)$, which would mean that the moment of inertia $J$ would be a linear map on $\text{Skew}(3)$. It can be shown that this matrix-to-matrix moment of inertia would take the form

$$J \Omega = \{V, \Omega\} = V \Omega + \Omega V,$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator of two matrices and where $V$ denotes the covariance matrix of the mass measure under consideration. $J$ then gives rise to a linear map $J$ on $\mathbb{R}^3$ by making the following diagram commute.

$$\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\text{ten}} & \text{Skew}(3) \\
J & \downarrow & J \\
\mathbb{R}^3 & \xrightarrow{\text{ten}} & \text{Skew}(3)
\end{array}$$

More precisely, for any $\omega \in \mathbb{R}^3$, $J \omega = \text{vec} J(\text{vec} \omega)$ and hence, using Lemma 6.1, the fact that, since $J = \{V, \cdot\}$, $J_{abcd} = V_{ac} \delta_{bd} + \delta_{ac} V_{bd}$, and the fact that $V$ is symmetric,

$$(J \omega)_i = \frac{1}{2} \epsilon_{abc} J_{abcd} \epsilon_{cmd} \omega_m = \frac{1}{2} \epsilon_{abc} (V_{ac} \delta_{bd} + \delta_{ac} V_{bd}) \epsilon_{cmd} \omega_m = \frac{1}{2} (\epsilon_{abc} V \epsilon_{cmd} + \epsilon_{abc} V \epsilon_{cmd}) \omega_m$$

$$= \frac{1}{2} \left( (\delta_{ac} \delta_{im} V_{ac} - \delta_{am} \delta_{ic} V_{ac} + \delta_{im} \delta_{bd} V_{bd} - \delta_{id} \delta_{bm} V_{bd}) \omega_m \right)$$

$$= \frac{1}{2} (V_{ai} \omega_i - V_{ai} \omega_i + V_{ai} \omega_i - V_{ai} \omega_i) = (\text{tr} V) \omega_i - (V \omega)_i,$$

i.e. indeed $J = (\text{tr} V)I - V$.

We now record a useful decomposition for the Eulerian velocity of a rigid body. Despite seeming quite innocuous, Proposition 3.3 below is quite important since it is later used to compute the linear and angular momentum of a rigid body. These computations are essential since they will in turn motivate the definition of the linear momentum and angular momentum densities.

**Proposition 3.3.** *(Decomposition of the Eulerian velocity of a rigid body)*

Let $(\mathcal{U}, \eta_{\overline{t}})$ be a rigid body. Then there exists constants $\vec{u}, \vec{\omega} \in \mathbb{R}^3$ such that, if $u$ denotes the Eulerian velocity and $\vec{x}$ denotes the center of mass of the rigid body, then we can decompose $u$ as $u = \vec{u} + \vec{\omega} \times (\cdot - \vec{x})$.

**Proof.** Since $\eta$ is a rigid motion we know from Proposition 2.11 that $\mathcal{D} u = 0$. Lemma 6.6 then tells us that there exists constants $\vec{u} \in \mathbb{R}^3$ and $\vec{\Omega} \in \text{Skew}(3)$ such that $u(\overline{t}) = \vec{u} + \vec{\Omega} \overline{t}$. So finally, for $\vec{\omega} := \text{vec} \vec{\Omega}$ and $\vec{u} := \vec{u} + \vec{\Omega} \vec{x}$ we have that $u = \vec{u} + \vec{\omega} \times (\cdot - \vec{x})$. $\square$

We now compute that mass, center of mass, covariance matrix, and moment of inertia of several simple rigid bodies. These computations serve several purposes.
The center of mass and covariance matrix are given by
\[
\bar{x} = \frac{1}{m} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{le_i}{2} m - \frac{-le_i}{2} m \right) \right) = 0 \quad \text{and} \quad V = \frac{1}{m} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{le_i}{2} \otimes \frac{le_i}{2} \right) \right) \frac{m}{2} + \left( \frac{-le_i}{2} \otimes \frac{-le_i}{2} \right) \frac{m}{2} = \frac{l^2}{4} e_n \otimes e_n
\]
and therefore the moment of inertia is
\[
J = m \left( \text{tr} V \right) I - V = m \left( \frac{l^2}{4} I - \frac{l^2}{4} e_n \otimes e_n \right) = \frac{ml^2}{4} (I - e_n \otimes e_n).
\]

**Example 3.5. (Idealized n-dumbbell)**
Consider the measure \( \nu = \frac{1}{n} \sum_{i=1}^{n} \frac{m}{2} \left( \delta_{e_i} + \delta_{-e_i} \right) \).

The center of mass and covariance matrix are given by
\[
\bar{x} = \frac{1}{m} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{le_i}{2} m + \frac{-le_i}{2} m \right) \right) = 0 \quad \text{and} \quad V = \frac{1}{m} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{le_i}{2} \otimes \frac{le_i}{2} \right) \right) \frac{m}{2} + \left( \frac{-le_i}{2} \otimes \frac{-le_i}{2} \right) \frac{m}{2} = \frac{l^2}{4n} \sum_{i=1}^{n} e_i \otimes e_i = \frac{l^2}{4n} I
\]
and therefore the moment of inertia is
\[
J = m \left( \text{tr} V \right) I - V = m \left( \frac{l^2}{4} I - \frac{l^2}{4n} I \right) = \frac{ml^2}{4n} - \frac{l^2}{n} I
\]

**Example 3.6. (Rod)**
Consider the measure \( \nu = \rho \mathcal{H}^1 \mathbb{L} \frac{1}{2} [-e_1, e_1] \) where \([-e_1, e_1] := \{ \theta (-e_1) + (1 - \theta) e_1 \mid \theta \in [0, 1] \} \). The mass and center of mass of the rod are given by
\[
M = \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho \mathcal{H}^1 (x) \, ds = \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho ds = \rho l \quad \text{and} \quad \bar{x} = \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho x \mathcal{H}^1 (x) \, ds = \frac{1}{\rho l} \left( \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho s ds \right) e_1 = 0.
\]
The covariance matrix is given by
\[
V = \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho x \otimes x \mathcal{H}^1 (x) \, ds = \left( \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho s^2 ds \right) e_1 \otimes e_1 = \frac{1}{\rho l^3} \int_{-\frac{1}{2} e_1}^{\frac{1}{2} e_1} \rho s^2 ds e_1 \otimes e_1 = \frac{l^2}{3} e_1 \otimes e_1.
\]
and hence the angular moment of inertia is \( J = \frac{r^2}{3} (I - e_1 \otimes e_1) \).

**Example 3.7.** (Sphere)
Consider the measure \( \nu = \mathcal{L}^n \mathbb{L} B (0, r) \). When evaluating integrals below, we will often use the following fact: given a map \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) preserving the Lebesgue measure, if the integrand \( f \) is odd under \( \Phi \), i.e. \( f \circ \Phi = -f \) and \( E \) is some (Lebesgue-measurable) set invariant under \( \Phi \), then \( \int_E f \, dx = 0 \).

First we compute the center of mass: by symmetry,
\[
\bar{x} = \int_{B(0, r)} x \, dx = 0.
\]

Now we compute the covariance matrix and moment of inertia. By symmetry,
\[
V_{ii} = \left( \int_{B(0, r)} x \otimes x \, dx \right)_{ii} = \int_{B(0, r)} x_i^2 \, dx = \frac{1}{n} \left( \int_{B(0, r)} x_1^2 \, dx + \cdots + \int_{B(0, r)} x_n^2 \, dx \right) = \frac{1}{n} \int_{B(0, r)} |x|^2 \, dx
\]
and
\[
V_{ij} = \left( \int_{B(0, r)} x \otimes x \, dx \right)_{ij} = \int_{B(0, r)} x_i x_j \, dx = 0 \text{ if } i \neq j.
\]

Now let us write \( \alpha_n := \mathcal{L}^n (B (0, 1)) \) and note that then \( \mathcal{H}^{n-1} (\partial B (0, 1)) = n \alpha_n \). We can then compute:
\[
\frac{1}{n} \int_{B(0, r)} |x|^2 \, dx = \frac{1}{n \mathcal{L}^n (B (0, r))} \int_0^r \left( \int_{\partial B (0, s)} |x|^2 \, d\mathcal{H}^{n-1} (x) \right) \, ds = \frac{1}{n \alpha_n r^n} \int_0^r \mathcal{H}^{n-1} (\partial B (0, s)) \, s^2 \, ds = \frac{\mathcal{H} (\partial B (0, 1))}{n \alpha_n r^n} \int_0^r s^{(n-1)+2} \, ds = \frac{n \alpha_n}{n \alpha_n r^n} \frac{r^{n+2}}{n+2} = \frac{r^2}{n+2}
\]

Therefore the covariance matrix is \( V = \frac{r^2}{n+2} I \). So finally the moment of inertia is
\[
J = \mathcal{L}^n (B (0, r)) \left( (\text{tr} V) I - V \right) = \alpha_n r^n \left( \frac{nr^2}{n+2} I - \frac{r^2}{n+2} I \right) = \frac{\alpha_n (n-1)}{n+2} r^{n+2} I.
\]

**Example 3.8.** (Cylinder)
Consider the measure \( nu = \mathcal{L}^3 \mathbb{L} C \) where
\[
C = B_2 (r) \times (-l, l) = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < r^2 \text{ and } |x_3| < l \}.
\]

Then the mass is
\[
M = \mathcal{L}^3 (C) = \mathcal{L}^2 (B_2 (r)) \mathcal{L}^1 ((-l, l)) = 2\pi r^2 l
\]
and by symmetry of \( C \) the center of mass is
\[
\bar{x} = \int_C x \, dx = \frac{1}{M} \int_{-l}^l \int_{B_2 (r)} x \, d(x_1, x_2) \, dx_3 = 0.
\]

Now let us compute the covariance matrix. Observe that by symmetry of \( C \), \( V_{ij} = \int_C x_i \otimes x_j \, dx = 0 \) when \( i \neq j \), and that
\[
V_{11} = V_{22} = \frac{1}{M} \int_{-l}^l \int_{B_2 (r)} x_1^2 \, d(x_1, x_2) = \frac{2l}{2\pi r l} \frac{1}{2} \int_{B_2 (r)} (x_1^2 + x_2^2) \, d(x_1, x_2) = \frac{1}{2\pi r^2} \int_0^r s^2 (2\pi s) \, ds = \frac{r^2}{4}
\]
whilst
\[
V_{33} = \frac{1}{M} \int_{-l}^l x_3^2 \, (\pi r^2) \, dx_3 = \frac{\pi r^2}{2\pi r l} \frac{2l^3}{3} = \frac{l^2}{3}.
\]

We have thus computed the covariance matrix to be
\[
V = \begin{pmatrix}
\frac{r^2}{4} & 0 & 0 \\
0 & \frac{r^2}{4} & 0 \\
0 & 0 & \frac{l^2}{3}
\end{pmatrix}.
\]
So finally, the moment of inertia is

\[ J = M \left( (\text{tr} V) I - V \right) = 2\pi r^2 I \begin{pmatrix} \frac{r^2}{2} + \frac{l^2}{3} \\ 0 \\ 0 \end{pmatrix} = 2\pi r^2 \begin{pmatrix} \frac{r^2}{4} + \frac{l^2}{3} & 0 & 0 \\ 0 & \frac{r^2}{4} & 0 \\ 0 & 0 & \frac{r^2}{2} \end{pmatrix} = \frac{\pi r^2}{6} \begin{pmatrix} 3r^2 + 4l^2 & 0 & 0 \\ 0 & 3r^2 + 4l^2 & 0 \\ 0 & 0 & 6r^2 \end{pmatrix} = \frac{M}{12} (3r^2 I + 4l^2 I_2 + 3r^2 e_3 \otimes e_3) \]

where \( I_2 := e_1 \otimes e_1 + e_2 \otimes e_2 \). In particular, if \( 4l^2 = 3r^2 \), then \( J = \frac{3M}{2} r^2 I \). It is worth contrasting this with the moment of inertia of a sphere of mass \( M \) and radius \( R \), which is \( \frac{2M}{5} r^2 I \).

Recall that the computations above are helpful since they provide building blocks that can later be used to construct more complicated mass measures – this will be essential in Section 3.2 when we characterize the moment of inertia of a sphere of mass \( M \).

Proposition 3.9. (Transformation of statistical functionals of a measure under rigid motions)

Let \( \nu \) be a finite Borel measure with finite second moment and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a rigid motion.

1. If \( T \) is a translation, i.e. \( T(x) = x + z \) for some \( z \in \mathbb{R}^n \), then the mass, center of mass, and covariance matrix of the measure transform as

\[ (M, \bar{x}, V) \to (M, \bar{x} + z, V) \]

i.e. \( (M, \bar{x} + z, V) \) are the statistical functionals of \( T_\# \nu \). In particular the moment of inertia is invariant under translations, i.e. \( J \to J \).

2. If \( T \) is a rotation, i.e. \( T(x) = Rx \) for some \( R \in O(n) \), then the mass, center of mass, and covariance matrix of the measure transform as

\[ (M, \bar{x}, V) \to (M, R\bar{x}, RVR^T) \]

i.e. \( (M, R\bar{x}, RVR^T) \) are the statistical functionals of \( T_\# \nu \). In particular the moment of inertia transforms as \( J \to RJR^T \).

Proof. (1) Denote by \( \mu \) the push-forward of \( \nu \) under \( T \), i.e. \( \mu := T_\# \nu \) such that

\[ d\mu(x + z) = d\nu(x) \]

Mass: performing the change of variables \( y = x + z \) we obtain

\[ \int_{E + z} d\mu(y) = \int_{E} d\mu(x + z) = \int_{E} d\nu(x) = M. \]

Center of mass: we perform the same change of variable \( y = x + z \) to compute

\[ \int_{E + z} yd\mu(y) = \int_{E} (x + z) d\nu(x) = \bar{x} + z. \]

Covariance matrix: since \( \bar{y} = \bar{x} + z \), performing the change of variables \( y = x + z \) yields \( y - \bar{y} = x - \bar{x} \) and hence

\[ \int_{E + z} (y - \bar{y}) \otimes (y - \bar{y}) d\mu(y) = \int_{E} (x - \bar{x}) \otimes (x - \bar{x}) d\nu(x) = V. \]

(2) Denote by \( \mu \) the push-forward of \( \nu \) under \( T \), i.e. \( \mu := T_\# \nu \) such that \( d\mu(Rx) = d\nu(x) \). To compute the mass, we perform the change of variables \( y = Rx \), noting that \( dy = dx \) since \( R \in O(n) \) and hence \( |\det R| = 1 \). Therefore we have

\[ \int_{RE} d\mu(y) = \int_{E} d\mu(Rx) = \int_{E} d\nu(x) = M. \]

Now we compute the center of mass, performing the same change of variables:

\[ \int_{RE} yd\mu(y) = \int_{E} Rxd\nu(x) = R\bar{x}. \]
Finally we compute the covariance matrix:
\[
\int_{R^E} (y - \bar{y}) \otimes (y - \bar{y}) \, d\mu(y) = \int_E R(x - \bar{x}) \otimes R(x - \bar{x}) \, d\nu(x) = RVR^T.
\]

Proposition 3.9 gives us a particularly simple rule for the transformation of statistical functionals under rigid motions that preserve the center of mass. This is recorded in Corollary 3.10 below.

**Corollary 3.10.** Under the transformation \( T(x) = R(x - \bar{x}) + \bar{x} \) for some \( R \in O(n) \), the mass, center of mass, and covariance matrix of a measure transform as

\[
(M, \bar{x}, V) \rightarrow (M, \bar{x}, RVR^T)
\]

In particular the moment of inertia transforms as \( J \rightarrow RJR^T \).

We continue establishing properties of statistical functionals of mass measures under various transformations. Having established how they transform under rigid motions we now record how they behave with respect to Cartesian products.

**Proposition 3.11.** (Statistical functionals of product measures)

Let \( \nu_1, \nu_2 \) be finite Borel measures with finite second moment on \( \mathbb{R}^{n_1}, \mathbb{R}^{n_2} \) respectively, with masses \( M_1, M_2 \), centers of masses \( \bar{x}_1, \bar{x}_2 \), and covariance matrices \( V_1, V_2 \). The measure \( \nu := \nu_1 \times \nu_2 \) on \( \mathbb{R}^{n_1+n_2} \) has mass

\[
M_1M_2, \text{ center of mass } (\bar{x}_1, \bar{x}_2), \text{ and covariance matrix } \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}.
\]

**Proof.** First we compute the mass: \( \nu(\mathbb{R}^{n_1+n_2}) = \nu(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \nu_1(\mathbb{R}^{n_1}) \nu_2(\mathbb{R}^{n_2}) = M_1M_2 \). Now we compute the covariance matrix, writing \( x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^{n_1+n_2} \) to obtain

\[
\int_{\mathbb{R}^{n_1+n_2}} x \otimes x \, d\nu(x) = \frac{1}{M_1M_2} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} (x_1, x_2) \, d\nu_1(x_1) \, d\nu_2(x_2) = \frac{1}{M_2} \int_{\mathbb{R}^{n_2}} (\bar{x}_1, x_2) \, d\nu_2(x_2) = (\bar{x}_1, \bar{x}_2).
\]

Finally, to compute the covariance matrix, we first note that since covariance matrices are invariant under translation, we may without loss of generality assume that \( \bar{x} = 0 \). Then

\[
\int_{\mathbb{R}^{n_1+n_2}} x \otimes x \, d\nu(x) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} (x_1, x_2) \otimes (x_1, x_2) \, d\nu_1(x_1) \, d\nu_2(x_2)
\]

\[
= \int_{\mathbb{R}^{n_2}} \left( \int_{\mathbb{R}^{n_1}} (x_1 \otimes x_1, x_2 \otimes x_1, x_1 \otimes x_2, x_2 \otimes x_2) \, d\nu_1(x_1) \right) \, d\nu_2(x_2) = \int_{\mathbb{R}^{n_2}} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \, d\nu_2(x_2) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}.
\]

\[\square\]

**3.2. Admissible moments of inertia.** The goal of this section is to characterize precisely when, given a positive symmetric matrix \( J \), it is the moment of inertia of some mass measure.

We begin with preliminary results from linear algebra used to formulate the conditions on the eigenvalues of \( J \) that will characterize admissible moments of inertia. First we record a result telling us to relate the spectrum of a covariance matrix with the spectrum of its moment of inertia.

**Lemma 3.12.** Let \( V \in \mathbb{R}^{n \times n} \) be symmetric and positive. Then \( S := (\text{tr} V) I - V \) is symmetric and positive. Moreover \( S \) and \( V \) have the same eigenspaces and, if \( \lambda = (\lambda_1, \ldots, \lambda_n) \) denotes the eigenvalues of \( V \) and \( \mu = (\mu_1, \ldots, \mu_n) \) denotes the corresponding (which is well-defined by item 1 above) eigenvalues of \( S \), then \( \mu = (C - I) \lambda \), where \( C \in \mathbb{R}^{n \times n} \) such that \( C_{ij} = 1 \) for all \( i, j \).

**Proof.** Clearly \( S \) is symmetric since \( V \) is. Now, since \( V \) is symmetric and positive, it has positive eigenvalues and is diagonal in some orthonormal basis \( \{v_i\}_{i=1}^n \), i.e. \( V = \sum_i \lambda_i v_i \otimes v_i \) for some \( \lambda_i > 0 \). Therefore, for any \( \xi \in \mathbb{R}^n \),

\[
S\xi \cdot \xi = \left( \sum_i \lambda_i \right) \|\xi\|^2 - \sum_i \lambda_i (v_i \cdot \xi)^2 = \sum_i \lambda_i \left( \|\xi\|^2 - (v_i \cdot \xi)^2 \right)
\]
where indeed, since $|v| = 1$ for each $i$, $|ξ|^2 - (v_i : ξ)^2 = |ξ|^2|v_i|^2 - (v_i : ξ)^2 \geq 0$ by Cauchy-Schwarz. Finally observe that

$$Sv_i = \left( \sum_{j \neq i} \lambda_j \right) v_i - \lambda_i v_i = \left( \sum_{j \neq i} \lambda_j \right) v_i$$

i.e. indeed $S$ and $V$ share the same eigenspaces, with moreover $μ = (C - I) \lambda$.

Since Lemma 3.12 established that there was a linear relationship between the spectrum of a covariance matrix and its associated moment of inertia we now observe, in Lemma 3.13 below, that this relationship is invertible and compute its inverse.

**Lemma 3.13.** Let $C \in \mathbb{R}^{n \times n}$ such that $C_{ij} = 1$ for all $i, j$. Then $C - I$ is invertible, with

$$(C - I)^{-1} = \frac{1}{n - 1} C - I.$$  

**Proof.** Since $C$ and $I$ commute, it is sufficient to show that $(C - I) \left( \frac{1}{n - 1} C - I \right) = I$. The key observation is that $C^2 = nC$. We may now compute directly that

$$(C - I) \left( \frac{1}{n - 1} C - I \right) = \frac{1}{n - 1} C^2 - C - \frac{1}{n - 1} C + I = \left( \frac{n}{n - 1} - 1 - \frac{1}{n - 1} \right) C + I = I.$$  

We now have the tools to establish the first of the two main results of this section, namely providing a necessary condition for a positive symmetric matrix to be the moment of inertia of some mass measure.

**Proposition 3.14.** (Admissible moments of inertia – necessity)

Let $ν$ be a finite Borel measure on $\mathbb{R}^n$ with finite second moment. Its moment of inertia is a positive symmetric $n$-by-$n$ matrix, and moreover if we denote by $μ \in \mathbb{R}^n$ its eigenvalues, then

$$(1) \quad μ_i \leq \frac{1}{n - 1} \sum_{j=1}^n μ_j \text{ for all } i = 1, \ldots, n$$

or equivalently

$$(2) \quad μ \in \mathbb{R}_+ \nabla^{n-1} = \{ sx \mid s \in \mathbb{R}_+, x \in \nabla^{n-1} \}$$

where $\mathbb{R}_+ := [0, \infty)$ and $\nabla^{n-1} = 1 - \Delta^{n-1}$, for $\Delta^{n-1}$ denoting the $(n - 1)$-simplex such that

$$\nabla^{n-1} := \left\{ x \in \mathbb{R}^n \mid x_i = 1 - θ_i = \sum_{j \neq i} θ_j \text{ for some } θ_j \geq 0 \text{ with } \sum_j θ_j = 1 \right\}.$$  

**Proof.** Suppose without loss of generality that $ν$ has unit mass. Let $V$ denote the covariance matrix of $ν$ and let $J$ denote the corresponding moment of inertia, i.e. $J = (\text{tr} V) I - V$. It follows from Lemma 3.12 that $J$ is symmetric, positive, has the same eigenspaces as $V$, and that if $λ$ and $μ \in \mathbb{R}^n$ denote the eigenvalues of $V$ and $J$ respectively, then $μ = (C - I) \lambda$ where $C \in \mathbb{R}^n$ such that $C_{ij} = 1$ for all $i, j$. Note that since $V$ is symmetric and positive, it follows that $λ \in \mathbb{R}_+^n$ and hence $μ \in (C - I) \mathbb{R}_+^n$. Now Lemma 3.13 says that $C - I$ is invertible with inverse $\frac{1}{n - 1} C - I$ and therefore $\left( \frac{1}{n - 1} C - I \right) μ \in \mathbb{R}_+^n$, i.e.

$$\frac{1}{n - 1} \sum_{j=1}^n μ_j - μ_i \geq 0 \text{ for all } i$$

so (1) holds. Now let us show that (1) $⇒$ (2). To do so, simply define

$$s := \frac{1}{n - 1} \sum_{j} μ_j \text{ and } θ := 1 - \frac{μ}{s}$$

where $1_i = 1$ for all $i$, and observe that then

- $s \geq 0$ since $J$ is positive and hence it’s eigenvalues $μ_i$ are positive,
• \( \theta \geq 0 \) since, by (1), \( s \geq 0 \), and
• \( \sum_1 \mu_i = n - \frac{1}{2} \sum_i \mu_i = n - (n - 1) = 1 \).

So indeed \( \mu = s(1 - \theta) \in \mathbb{R}_+ \mathbb{V}^{n-1} \). Finally we show that \( (2) \Rightarrow (1) \). This is immediate since by assumption \( \mu = s(1 - \theta) \) for some \( s \geq 0 \) and some \( \theta \in \mathbb{D}^{n-1} \), and therefore \( \sum_i \mu_i = s(n - 1) \), i.e. \( s = \frac{1}{n-1} \sum_i \mu_i \). We can thus conclude that, since \( \theta_i \leq 1 \), \( \mu_i = s(1 - \theta_i) \leq s = \frac{1}{n-1} \sum_i \mu_i \).

The remainder of this section is now devoted to proving that the necessary condition for a positive symmetric matrix to be the moment of inertia of some mass measure recorded in Proposition 3.14 above is actually sufficient. To do so means being able to, given any positive symmetric matrix \( J \equiv \rho \mathcal{H}^1 \mathbb{L}^2 \mu \), to actually generate any prescribed moment of inertia.

Proof. If \( \lambda = 0 \) then define a ‘point-mass’ measure \( \nu = M \delta_0 \) such that indeed
\[
\nu(R^n) = M, \quad \mathbb{E}_\nu [x] = 0, \quad \text{and} \quad \mathbb{V}_\nu [x] = \mathbb{E}_\nu [x \otimes x] = 0.
\]
If \( \lambda > 0 \) then define \( \nu := \lambda \mathcal{H}^1 \mathcal{L}^2 [-e_1, e_1] \) where \( l \equiv \sqrt{\lambda} \) and \( \rho := \frac{M}{l} \). We may thus compute, as in Example 3.6, and obtain
\[
\nu(R^n) = \rho l = M, \quad \mathbb{E}_\nu [x] = 0, \quad \text{and} \quad \mathbb{V}_\nu [x] = \frac{l^2}{3} e_1 \otimes 3_1 = \lambda e_1 \otimes e_1.
\]

Using Lemma 3.15 immediately above and Proposition 3.11 we now show that we can construct mass measures corresponding to arbitrary diagonal covariance matrices.

Lemma 3.16. (Existence of mass measures with prescribed diagonal covariance matrix)
For any \( M > 0 \) and any \( \lambda_1, \ldots, \lambda_n \geq 0 \) there exists a Borel measure \( \nu \) on \( \mathbb{R}^n \) with mass \( M \), center of mass \( 0 \) and covariance matrix \( \lambda e_1 \otimes e_1 \).

Proof. This follows immediately from combining Lemma 3.15 and Proposition 3.11. For any \( M > 0 \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \), by Lemma 3.15 there are Borel measures \( \nu_i \), \( i = 1, \ldots, n \), on \( \mathbb{R}^1 \) with masses \( M^{1/n} \), centers of mass \( 0 \) and covariance matrices \( \lambda_i \) (\( i \in \mathbb{R}^{1 \times 1} = \mathbb{R} \)). Therefore, by Proposition 3.9 the measure \( \nu := \nu_1 \otimes \cdots \otimes \nu_n \) has the desired statistical functionals.

Combing Lemma 3.16 with Lemma 3.12 allows us to deduce that we can construct mass measures corresponding to arbitrary diagonal moment of inertia.

Lemma 3.17. (Existence of mass measures with prescribed diagonal moment of inertia)
For any \( M > 0 \) and any \( \mu \in \mathbb{R}_+ \mathbb{V}^{n-1} \) there exists a Borel measure \( \nu \) on \( \mathbb{R}^n \) with finite second moment, mass \( M \), center of mass \( 0 \) and moment of inertia \( \mu \).

Proof. Let \( \lambda := \frac{1}{M} \left( \frac{1}{n-1} (C - I) \right) \mu \). Note that part (1) of Proposition 3.14 tells us precisely that \( \lambda \in \mathbb{R}_+ \). By Lemma 3.16 we therefore know that there exists a Borel measure \( \nu \) with mass \( M \), center of mass \( 0 \) and covariance matrix \( \lambda_1, \ldots, \lambda_n \). Since the moment of inertia is \( J = M ((\text{tr} V) I - V) \), Lemma 3.12 tells us that \( J \) is diagonal with eigenvalues \( M (C - I) \lambda \). Finally, since Lemma 3.13 tells us that \( (C - I)^{-1} = \frac{1}{n-1} C - I \), it follows that the eigenvalues of \( J \) are indeed \( \mu \).

We now have all the tools to prove the sufficiency of the conditions for admissibility of a moment of inertia introduced in Proposition 3.14. We do so to conclude this section.
Proposition 3.18. (Admissible moments of inertia – sufficiency)
Let $M > 0$ and let $J$ be a positive symmetric $n$-by-$n$ matrix with eigenvalues $\mu_1, \ldots, \mu_n$ satisfying

$$\mu_i \leq \frac{1}{n-1} \sum_{j=1}^{n} \mu_j \text{ for every } i.$$  

Then there exist a Borel measure $\nu$ with mass $M$, center of mass 0, and angular moment of inertia $J$.

Proof. Since $J$ is symmetric and positive, there exists a rotation matrix $R \in O(n)$ such that

$$J = R \, \text{diag} (\mu_1, \ldots, \mu_n) \, R^T.$$

Since, by Proposition 3.14, $\mu \in \mathbb{R}_+ \nabla^{n-1}$ it follows from Lemma 3.17 that there exists a Borel measure $\bar{\nu}$ with mass $M$, center of mass 0 and moment of inertia $\text{diag} (\mu_1, \ldots, \mu_n)$. Finally Corollary 3.10 tells us that after applying the transformation $T(x) := R(x - \bar{x}) + \bar{x}$ the measure $\nu := T_\# \bar{\nu}$ has mass $M$, center of mass 0 and moment of inertia $R \, \text{diag} (\mu_1, \ldots, \mu_n) \, R^T = J$.

Putting Proposition 3.14 and Proposition 3.18 together, we have proved the theorem below.

Theorem 3.19. (Admissible moments of inertia)
Let $J$ be a positive symmetric $n$-by-$n$ matrix. There exists a finite Borel measure $\nu$ on $\mathbb{R}^n$ with moment of inertia $J$ if and only if the eigenvalues $\mu = (\mu_1, \ldots, \mu_n)$ of $J$ satisfy

$$\mu_i \leq \frac{1}{n-1} \sum_{j=1}^{n} \mu_j \text{ for every } i.$$  

3.3. Linear and angular momentum. In this section we define the linear and angular momentum associated with a rigid body and compute their values for a rigid body.

Definition 3.20. (Linear and angular momentum)
Let $(\Omega, \eta)$ be a continuum with Eulerian mass measure $\nu$. For every Borel set $E \subseteq \Omega (t)$ we define

(1) its linear momentum $\mathcal{P} (E) := \int_E u \, d\nu_t$ and

(2) if $n = 3$, its angular momentum about a point $z \in \mathbb{Z}^n$ is defined to be

$$\mathcal{L}_z (E) := \int_E (x - z) \times u_t (x) \, d\nu_t (x).$$

In particular, its angular momentum about its center of mass is simply called its angular momentum and denoted $\mathcal{L} (E)$.

Having defined linear and angular momentum, we compute their values for a rigid body.

Proposition 3.21. (Linear and angular momentum of a rigid body)
Let $(\mathcal{U}, \eta)$ be a rigid body. Then, using the notation of Proposition 3.3 for $\bar{u}$ and $\bar{\omega}$ and using $M$ and $J$ to denote respectively the mass and moment of inertia of $\mathcal{U} (t)$, we have

$$\mathcal{P} (\mathcal{U} (t)) = M \bar{u} \text{ and } \mathcal{L} (\mathcal{U} (t)) = J \cdot \bar{\omega}.$$  

Proof. In light of Proposition 3.3 we may immediately compute the linear momentum to be

$$\mathcal{P} (\mathcal{U} (t)) = \int_{\mathcal{U} (t)} u \, d\nu_t = \int_{\mathcal{U} (t)} (\bar{u} + \omega \times (x - \bar{x})) \, d\nu_t = M \bar{u} + \omega \times \left( \int_{\mathcal{U} (t)} x \, d\nu_t - M \bar{x} \right) = M \bar{u}.$$  

Using the ‘vectorized’ version of the epsilon-delta identity in Lemma 6.2 we may now compute the angular momentum to be

$$\mathcal{L} (\mathcal{U} (t)) = \int_{\mathcal{U} (t)} (x - \bar{x}) \times u \, d\nu_t = \int_{\mathcal{U} (t)} ((x - \bar{x}) \times \bar{u} + (x - \bar{x}) \times (\omega \times (x - \bar{x}))) \, d\nu_t$$

$$= 0 + \int_{\mathcal{U} (t)} \left( |x - \bar{x}|^2 \omega - (\omega \cdot (x - \bar{x})) (x - \bar{x}) \right) \, d\nu_t = J \cdot \omega.$$

Proposition 3.21 comes in handy later since it motivates the definition of a linear momentum density and a angular momentum density.
4. Conservation laws

The goal of this section is two-fold:
(1) introduce the integral balance laws corresponding to the conservation of mass, linear and angular momentum, and energy, and
(2) derive the corresponding local versions of these balance laws.

To be more precise, we discuss the conservation of mass in Section 4.1, the conservation of microinertia in Section 4.2, the conservation of linear and angular momentum in Section 4.3. We conclude Section 4 with a brief discussion of boundary conditions in Section 4.5.

4.1. Mass. In this section we define what it means for a mass measure to be conserved and we derive the associated local conservation law. We also use the conserved mass of rigid body to obtain a useful characterization of the kinematic and dynamics variables describing a rigid body.

Definition 4.1. (Conservation of mass)
Let \((\Omega, \eta)\) be a continuum. An Eulerian mass measure \(\nu\) satisfying \(\nu_t = (\eta_t)_\# \nu_0\) is called a conserved Eulerian mass measure.

Having defined what it means for mass to be conserved we derive the associated local conservation law, which turns out to be the well-known continuity equation.

Proposition 4.2. (Local conservation of mass - continuity equation)
Let \((\Omega, \eta)\) be a continuum with Eulerian velocity \(u\). If a conserved Eulerian mass measure is absolutely continuous with respect to the Lebesgue measure, then its Eulerian mass density \(\rho\) satisfies
\[
\partial_t \rho + \nabla \cdot (\rho u) = 0.
\]
In other words: as a consequence of conservation of mass, the Eulerian mass density satisfies the continuity equation.

Proof. We simply use Reynolds’ transport theorem, i.e. Theorem 2.7, to take a time derivative of the equation of conservation of mass for an arbitrary set. Indeed, if we let \(\mathcal{U}_0 \subseteq \Omega_0\) be any open set, then
\[
0 = \frac{d}{dt} \mathcal{M}(\mathcal{U}_0) = \frac{d}{dt} \mathcal{M}(\mathcal{U}(t)) = \frac{d}{dt} \int_{\mathcal{U}_0} \rho = \int_{\mathcal{U}(t)} \partial_t \rho + \nabla \cdot (\rho u).
\]
Therefore, since \(\mathcal{U}_0\) was an arbitrary subset of the Lagrangian domain and since \(\eta_t\) are diffeomorphisms (such that we can go back and forth between Lagrangian and Eulerian coordinates), the integrand above must vanish everywhere in the Eulerian domain. \(\square\)

We now record a result relating the fact that a flow is locally volume preserving with the absolute continuity of its conserved mass measure. This result will be used in Proposition 4.5 below, which is itself used to justify the definition of micropolar continua.

Proposition 4.3. (Locally volume-preserving flows preserve absolute continuity of the Eulerian mass measure)
Let \(\eta\) be a locally volume-preserving flow map and let \(\nu\) be the conserved Eulerian mass measure. Suppose that \(\nu\) is initially absolutely continuous (with respect to the Lebesgue measure). Then \(\nu\) is absolutely continuous for all time, and moreover we have an explicit representation for the Eulerian mass density in terms of the initial density and the flow map, namely: \(\rho_t = \rho_0 \circ \eta_t^{-1}\).

Proof. Since \(\eta\) is locally volume-preserving and \(\nu\) is conserved by the flow, both sides of the ‘inequality’ \(\nu_0 \ll \mathcal{L}^n\) are preserved under pushforwards along the flow, i.e.: \(\nu_t = (\eta_t)_\# \nu_0 \ll (\eta_t)_\# \mathcal{L}^n = \mathcal{L}^n\). Moreover
\[
(\rho_t d\mathcal{L}^n)(E) = dv_t(E) = dv_0 \left( \eta_t^{-1}(E) \right) = \int_{\eta_t^{-1}(E)} \rho_0(y) dy = \int_E \rho_0(\eta_t^{-1}(x)) dx = \left( \rho_0 \circ \eta_t^{-1} \right)(d\mathcal{L}^n)(E)
\]
i.e. indeed \(\rho_t = \rho_0 \circ \eta_t^{-1}\), where (*) holds since \(\eta\) is locally volume-preserving, and hence by Lemma 2.14 \(|\det \nabla \eta| = 1\). \(\square\)

We now record another result having to do with the densities of conserved mass measures which allows us to translate between the Lagrangian and Eulerian mass densities.
Let \( x \) that and on the other hand, upon using the change of variables mass measure with mass density \( \nu \).

Since \( \eta \) is a rigid body with conserved Eulerian mass measure \( E \), we have in particular that \( \nu \) is rigid motion and so by Lemma 2.14.

Proof. Let \( v_0 \in \Omega_0 \), let \( r > 0 \), and let \( t \geq 0 \). On one hand

\[
\mu_t (B(y_0, r)) = \int_{B(y_0, r)} \sigma_t (y) dy
\]

and on the other hand, upon using the change of variables \( x = \eta_t (y) \) we see that

\[
\nu_t (\eta_t (B(y_0, r))) = \int_{B(y_0, r)} \rho_t (x) dx = \int_{B(y_0, r)} (\rho_t \circ \eta_t) (y) \det \nabla \eta_t (y) dy.
\]

Since \( \nu_t = (\eta_t)_\# \mu_t \), we have in particular that \( \nu_t (\eta_t (B(y_0, r))) = \mu_t (B(y_0, r)) \), and hence

\[
\int_{B(y_0, r)} \sigma_t (y) dy = \int_{B(y_0, r)} (\rho_t \circ \eta_t) (y) \det \nabla \eta_t (y) dy.
\]

So finally, dividing both sides of the equation immediately above by \( L^n (B(y_0, r)) \) and sending \( r \downarrow 0 \) we obtain:

\[
\sigma_t (y_0) = \lim_{r \downarrow 0} \int_{B(y_0, r)} \sigma_t (y) dy = \int_{B(y_0, r)} (\rho_t \circ \eta_t) (y) \det \nabla \eta_t (y) dy = (\rho_t \circ \eta_t) (y_0) \det \nabla \eta_t (y_0).
\]

We conclude this section with a characterization of the kinematic and dynamic descriptors of a rigid body, i.e. its flow map \( \eta \) and its Eulerian velocity \( u \) respectively, provided this rigid body has a conserved mass measure.

Proposition 4.5. (Canonical representation of rigid motions via their Eulerian mass measures)

If \( \eta \) is a rigid body with conserved Eulerian mass measure \( \nu \), then

\[
\eta(t, y) = \bar{x}(t) + R(t)(y - \bar{x}(0)) \quad \text{and} \quad u(t, x) = \dot{u}(t) + \Omega(t)(x - \bar{x}(t))
\]

where \( \bar{x} : [0, \infty) \to \mathbb{R}^n \), \( V : [0, \infty) \to \text{Sym}(n) \), and \( R : [0, \infty) \to O(n) \) are given by, for every \( t \geq 0 \),

\[
\bar{x}(t) = \mathbb{E}_{\nu_t} [x] = \int_{\mathbb{U}(t)} x d\nu_t (x), \quad V(t) = \mathbb{V}_{\nu_t} [x] = \int_{\mathbb{U}(t)} (x - \bar{x}(t)) \otimes (x - \bar{x}(t)) d\nu_t (x), \quad \text{and} \quad R(t) = V(t)^{1/2} V(0)^{1/2}
\]

and \( \bar{u} : [0, \infty) \to \mathbb{R}^n \) and \( \Omega : [0, \infty) \to \text{Skew}(n) \) are given by, for every \( t \geq 0 \),

\[
\bar{u}(t) = \dot{\bar{x}}(t) \quad \text{and} \quad \Omega(t) = \dot{R}(t) R(t)^T = \left( V(t)^{1/2} \right)' V(t)^{-1/2}.
\]

Proof. First we show that \( \eta = \bar{x} + R(\cdot - \bar{x}(0)) \). Since \( \eta \) is a rigid motion, we have by Proposition 2.11 that \( \eta (t, y) = z (t) + R(t) y \) for some \( z : [0, \infty) \to \mathbb{R}^n \) and some \( R : [0, \infty) \to O(n) \). The key computation is:

\[
\frac{1}{\mathcal{M}(U(t))} \int_{U(t)} x \rho_t (x) dx = \frac{1}{\mathcal{M}(U(t))} \int_{U(t)} \eta(y, t) \rho_t (\eta(y, t), t) dy \overset{(1)}{=} \frac{1}{\mathcal{M}(U(t))} \int_{U(t)} (z(t) + R(t) y) \rho_0 (y) dy
\]

which relies in (1) on the fact that \( \eta \) is a rigid motion and so by Lemma 2.14 \( \det \nabla \eta = 1 \), and in (2) on the fact that \( \rho_t = \rho_0 \circ \eta_t \) by Proposition 4.3. So finally, where the computation above comes into play in (+), we obtain that:

\[
\tau = \mathbb{E}_{\nu_0} (x) = \mathbb{E}_{\nu_0} (z + R y) = z + R \mathbb{E}_{\nu_0} (y) = z + R \tau (0).
\]

Now we show that \( R = V^{1/2} V_0^{-1/2} \). To do this, we first compute how to express the covariance matrix at a time \( t \) in terms of the initial covariance matrix. Using the same change of variable as above, and observing that \( x - \bar{x} = \eta - \eta = R (y - \bar{x}_0) \), we obtain that, using Lemma 6.5,

\[
V = \mathbb{V}_{\nu_t} \left( (x - \bar{x}) \otimes (x - \bar{x}) \right) = \mathbb{V}_{\nu_0} \left( R(y - \bar{x}_0) \otimes R(y - \bar{x}_0) \right) = R \mathbb{V}_{\nu_0} \left( (y - \bar{x}_0) \otimes (y - \bar{x}_0) \right) R^T = RV_0 R^T.
\]
Now all that is left to do is to solve the conjugacy equation $V = RV_0R^T$ for $R$. We make the educated guess $R := V^{1/2}V_0^{-1/2}$, noting that the square root of $V$ is well-defined since $V$ is symmetric and positive-definite, and observe that then:

$$RV_0R^T = V^{1/2} \left( V_0^{-1/2}V_0 \left( V_0^{-1/2} \right)^T \right) \left( V^{1/2} \right)^T = V^{1/2}V^{1/2} = V$$

where we have used that for $S$ symmetric and positive-definite, $(S^{1/2})^T = (S^T)^{1/2} = S^{1/2}$.

Now we show that $\Omega = \dot{R}R^{-1}$. Since the flow map is

$$\eta_t(y) = \overline{x}(t) + R(t)(y - \overline{x}(0))$$

it follows that the inverse flow map is

$$\eta_t^{-1}(x) = \overline{x}(0) + R(t)^{-1}(x - \overline{x}(t)).$$

We can therefore compute the Lagrangian and Eulerian velocities to be

$$v(y, t) = \frac{d}{dt}\eta_t(y) = \dot{x} + \dot{R}(t)(y - \overline{x}(0))$$

and

$$u(x, t) = v(\eta_t^{-1}(x), t) = \dot{x}(t)\overline{u}(t) + \dot{R}(t)R(t)^{-1}(x - \overline{x}(t)) = \ddot{u}(t) + \Omega(t)(x - \ddot{x}(t)).$$

Finally we show that $\Omega = (V^{1/2})'V^{-1/2}$. We proceed as above, but using $V$ instead of $R$. Indeed, since

$$\eta_t = \overline{x} + V^{1/2}V_0^{-1/2}(\cdot - \overline{x}_0)$$

it follows that

$$\eta_t^{-1} = \overline{x}_0 + \left( V^{1/2}V_0^{-1/2} \right)^{-1}(\cdot - \overline{x}).$$

Therefore

$$v = \overline{x} + \left( V^{1/2}V_0^{-1/2} \right)'$$

and

$$u = \dot{x} + \left( V^{1/2}V_0^{1/2} \right)' \left( V^{1/2}V_0^{-1/2} \right)^{-1}(\cdot - \overline{x}) = \dot{x} + (V^{1/2})'V^{-1/2}(\cdot - \overline{x}).$$

\[\square\]

Proposition 4.5 above helps us motivate the definition of a micropolar continuum provided in Definition 2.17. Indeed, Proposition 4.5 tells us that a rigid motion may be fully characterized by the behaviour of its center of mass $\overline{x}(t)$ and some rotation matrix $R(t)$, and therefore it stands to reason that we would define the motion of a micropolar continuum to be determined by a flow map $\eta$ and a microrotation map $Q$: $\eta$ plays the role of $\overline{x}$ since it tracks the translational motion of the microstructure, and $Q$ plays the role of $R$ since it tracks the rotational motion of the microstructure.

4.2. Microinertia. In this section we define the microinertia of a micropolar continuum, define what it means for microinertia to be conserved, and finally derive the associate local conservation law. First we recall that we have characterized the space of admissible moments of inertia.

**Definition 4.6. (Admissible moments of inertia)**

We denote by $\mathcal{I}(n)$ the set of admissible moments of inertia, i.e. in light of Theorem 3.19,

$$\mathcal{I}(n) := \left\{ J \in \mathbb{R}^{n \times n} \mid J \geq 0, J = J^T, \text{ and its eigenvalues } \mu_1, \ldots, \mu_n \text{ satisfy } \mu_i \leq \frac{1}{n - 1} \sum_{j=1}^{n} \mu_j \text{ for all } i \right\}.$$

We now define the microinertia of a micropolar continuum, which is nothing more than a function defined along the flow taking values in the space of admissible moments of inertia.

**Definition 4.7. (Microinertia)**

Let $(\Omega_0, \eta, Q)$ be a micropolar continuum.

1. A Lagrangian function defined along the flow with codomain $\mathcal{I}(n)$, i.e. a map

$$i : \Omega_0 \times [0, \infty) \to \mathcal{I}(n),$$

is also called a Lagrangian microinertia density.
Given a Lagrangian mass measure $\mu$ and a Lagrangian microinertia density $i$, the Lagrangian measure defined along the flow $\gamma := i\mu$ is also called a Lagrangian microinertia measure, and we say that $\gamma$ is subordinate to $\mu$ to mean that $\gamma$ is absolutely continuous with respect to $\mu$. Moreover, if $\mu$ has an associated Lagrangian mass density $\sigma$ (i.e. $\mu \ll \mathcal{L}^n$ and $\sigma = \frac{d\mu}{d\mathcal{L}^n}$) then $I := i\sigma$ is called a Lagrangian microinertia.

(3) An Eulerian function defined along the flow with codomain $\mathcal{I}(n)$ is also called an Eulerian microinertia density.

(4) Given an Eulerian mass measure $\nu$ and an Eulerian microinertia density $j$, the Eulerian measure defined along the flow $\lambda$, defined via $\lambda_t := j_t \nu_t$ for all $t \geq 0$, is also called an Eulerian microinertia measure, and we say that $\lambda_t$ is subordinate to $\nu$ to mean that $\lambda_t$ is absolutely continuous with respect to $\nu_t$ for every $t \geq 0$. Moreover, if $\nu$ has an associated Eulerian mass density $\rho$ (i.e. $\nu_t \ll \mathcal{L}^n$ and $\rho_t = \frac{d\nu_t}{d\mathcal{L}^n}$ for all $t \geq 0$) then $J := j\rho$ is called an Eulerian microinertia.

In Definition 4.7 above we make a careful distinction between the microinertia density and the microinertia. However in the sequel we will almost exclusively devote our attention to incompressible flows, in which case the microinertia density and the microinertia are the same up to a constant factor of the mass density $\rho$. Nonetheless, it is important to make this distinction in order for the foundation of micropolar continuum mechanics laid so far to also be useful when it is used to investigate compressible micropolar fluids.

We now define what it means for the microinertia density to be conserved. This definition is inspired from the way in which the moment of inertia transforms under rigid motions that preserve the center of mass, as recorded in Corollary 3.10.

**Definition 4.8. (Conservation of a microinertia density)**

Let $(\Omega_0, \eta, Q)$ be a micropolar continuum

(1) A Lagrangian microinertia density $i$ satisfying $i = Q \eta_0 Q^T$, i.e.

$$i(t, y) = Q(t, y) i(0, y) Q^T(t, y) \quad \text{for all } (t, y) \in [0, \infty) \times \Omega_0$$

is called a conserved Lagrangian microinertia density.

(2) An Eulerian microinertia density $j$ is called a conserved Eulerian microinertia density if $j_t = i_t \circ \eta_t^{-1}$ for all $t \geq 0$ for some conserved Lagrangian microinertia density $i$.

Having defined what it means for microinertia density to be conserved we derive the associate local conservation law.

**Proposition 4.9. (Local conservation of a microinertia density)**

Let $(\Omega_0, \eta, Q)$ be a micropolar continuum with Eulerian velocity $u$, Eulerian angular velocity tensor $\Omega$ and conserved Eulerian microinertia density $j$. Then

$$\partial_t j + (u \cdot \nabla) j - [\Omega, j] = 0.$$

**Proof.** Let $i$ be the Lagrangian microinertia density corresponding to $j$, i.e. $i = j \circ \eta$. Then, by conservation of microinertia,

$$j \circ \eta = i = Q \eta_0 Q^T.$$

Upon taking a time derivative, using Proposition 2.6 to compute $\frac{d}{dt} (j \circ \eta)$ and denoting by $\Theta$ the Lagrangian angular velocity tensor $\Theta := \Omega \circ \eta = (\partial_t Q) Q^{-1}$ we have that

$$((\partial_t + u \cdot \nabla) j) \circ \eta = \partial_t Q \eta_0 Q^T + Q \eta_0 \partial_t Q^T = (\partial_t Q Q^{-1}) (Q \eta_0 Q^T) + (Q \eta_0 Q^T) \left(Q^{-T} \partial_t Q^T\right)$$

$$= \Theta (j \circ \eta) + (j \circ \eta) \Theta^T.$$

So finally, precomposing by $\eta^{-1}$ on both sides and recalling that $\Theta$ and $\Omega$ are anti-symmetric we obtain that

$$\partial_t j + (u \cdot \nabla) j = (\Theta \circ \eta^{-1}) j + j (\Theta \circ \eta^{-1})^T = \Omega j - j\Omega.$$

$\square$

It is worth noting that the differential operator $\partial_t + u \cdot \nabla - [\Omega, \cdot]$, which appears in Proposition 4.9 and will appear throughout the sequel whenever the microinertia is involved, is an analog of the well-known advection operator $\partial_t + u \cdot \nabla$. Indeed: while the advection operator $\partial_t + u \cdot \nabla$ takes into account the change in a quantity defined along the flow due to the advection by the flow map, a heuristic explanation made
precise in Proposition 2.6, the operator $\partial_t + u \cdot \nabla - [\Omega, \cdot]$ also takes into account the rotation due to the microrotation map. We will thus refer to this operator as an *advection-rotation* operator.

We now note that, as a consequence of Proposition 4.2 and Proposition 4.9 which provide us with the local conservations of mass and microinertia density respectively, we may now derive the local conservation law satisfied by the microinertia.

**Corollary 4.10.** *(Local conservation of the microinertia)*

Let $(\Omega_0, \eta, Q)$ be a micropolar continuum with Eulerian velocity $u$ and Eulerian angular velocity tensor $\Omega$. Let $\rho$ be an Eulerian mass density which is conserved and let $j$ be a conserved Eulerian microinertia density. Then the microinertia $J := j\rho$ satisfies

$$\partial_t J + \nabla \cdot (J \otimes u) - [\Omega, J] = 0.$$ 

**Proof.** In light of Proposition 4.2 which says that $D_t^u j = [\Omega, j]$, Proposition 4.9 which says that $D_t^\rho \rho = 0$, and Lemma 6.7 which provides a “Leibniz Rule” for material derivatives, this result is the consequence of a direct computation:

$$\partial_t J + \nabla \cdot (J \otimes u) = D_t^\rho J = D_t^\rho (j\rho) = (D_t^u j)\rho + j(D_t^\rho \rho) = [J, \rho] = [J, \Omega].$$

\[ \square \]

### 4.3. Momenta and stresses.

In this section we define the linear momentum measure and angular momentum measure associated with a conserved mass measure, state what it means for linear and angular momentum to be conserved — introducing the stress and couple stress tensor in the process, and conclude with the derivation of the associated local conservation law. First we define the linear momentum measure and angular momentum measure.

**Definition 4.11.** *(Linear momentum measure, linear momentum density)*

Let $(\Omega_0, \eta)$ be a continuum, let $\nu$ be an Eulerian mass measure, and let $u$ denote the Eulerian velocity of the continuum. We call $\nu u$ a linear momentum measure. Moreover, if $\rho$ is the Eulerian mass density of $\nu$, i.e. $d\nu = \rho d\mathcal{L}^n$, then we call $\rho u$ a linear momentum density.

**Definition 4.12.** *(Angular momentum measure, angular momentum density)*

Let $n = 3$, let $(\Omega_0, \eta, Q)$ be a micropolar continuum, let $\nu$ be an Eulerian mass measure, let $j$ be an Eulerian microinertia density and let $\omega$ be the Eulerian angular velocity of the continuum. We call $j\omega \nu$ an angular momentum measure. Moreover, if $\rho$ is the Eulerian mass density of $\nu$, i.e. $d\nu = \rho d\mathcal{L}^n$, and $J := \rho i$ is the associated Eulerian microinertia then we call $J\omega$ an angular momentum density.

With Definition 4.11 and Definition 4.12 in hand we are now ready to define what it means for linear and angular momentum to be conserved. Prior to doing so we introduce the notion of a *physical* micropolar continuum. This is simply a convenient way to combine together *kinematic* and *physical* information. The kinematic information, namely the flow map and the microrotation map, is already built in the definition of a micropolar continuum, and so when defining a physical micropolar continuum we add in physical information, namely postulating the existence of a conserved mass measure and a conserved microinertia density.

**Definition 4.13.** *(Physical micropolar continuum)*

A *physical micropolar continuum* is a tuple $(\Omega_0, \eta, Q, \nu, j)$ such that

1. $(\Omega_0, \eta, Q)$ is a micropolar continuum,
2. $\nu$ is a conserved Eulerian mass measure, and
3. $j$ is a conserved Eulerian microinertia density.

At last we are equipped to define what it means for linear and angular momentum to be conserved. Note that this introduces the notion of a *stress tensor* and a *couple stress tensor*. The postulation that such tensors exist is a core tenet of rational continuum mechanics. Physically, these stress and couple stress tensors are manifestations of Newton’s third law: “Every action creates an equal and opposite reaction”, which in this context means that they encode how the fluid reacts to forces and torques induced by the neighbouring fluid.

**Definition 4.14.** *(Balance of momenta, stress and couple stress tensor, and external force and torque)*

Let $n = 3$, let $(\Omega_0, \eta, Q, \nu, j)$ be a physical micropolar continuum, let

- $u$ denote the Eulerian velocity,
• $\omega$ denote the Eulerian angular velocity,
• $\rho$ denote the Eulerian mass density, i.e. $d\nu = \rho dE$,
• $J = \rho j$ denote the Eulerian microinertia,
• $T, M$ be Eulerian functions defined along the flow with codomain $\mathbb{R}^{3 \times 3}$ called the stress tensor and couple stress tensor respectively,
• $f, \tau$ be Eulerian functions defined along the flow with codomain $\mathbb{R}^3$ called the external force and external torque respectively.

If, for every $\mathcal{U}_0 \subset \subset \Omega_0$,
\[
\frac{d}{dt} \int_{\mathcal{U}(t)} \rho u \, dx = \int_{\partial \mathcal{U}(t)} T n \, dx + \int_{\mathcal{U}(t)} f \, dx
\]
and
\[
\frac{d}{dt} \left( \int_{\mathcal{U}(t)} x \times \rho u \, dx + \int_{\mathcal{U}(t)} J \omega \, dx \right) = \int_{\partial \mathcal{U}(t)} x \times (T n) \, dx + \int_{\partial \mathcal{U}(t)} M n \, dx + \int_{\mathcal{U}(t)} x \times f \, dx + \int_{\mathcal{U}(t)} \tau \, dx
\]
where $\mathcal{U}(t) := \eta_t(\mathcal{U}_0)$ for every $t \geq 0$, then we say that the physical micropolar continuum is governed by $(T, M)$ subject to $(f, \tau)$. Moreover, the two integral equations above are referred to as the balance of linear momentum and the balance of angular momentum respectively.

Note that in the definition of the balances of linear and angular momentum above we restrict the integral balances to subsets of the continuum satisfying $\mathcal{U}_0 \subset \subset \Omega_0$, i.e. staying away from the boundary of the continuum. This is done because the stress tensor $T$ and the couple stress tensor $M$ are manifestations of Newton’s third law. Indeed, as mentioned earlier, these tensors encode the fact that the continuum reacts to forces and torques induced at a point by the continuum present in a neighbourhood of that point. In particular, if $\partial \mathcal{U}_0 \cap \partial \Omega_0 \neq \emptyset$ then the balances of linear and angular momentum must be modified slightly from their respective versions provided in Definition 4.14 above. This is done in Section 4.5.

Lemmas 4.15 and 4.16 below are useful intermediate computations which are consequences of the local conservation of mass and microinertia respectively, as well as the “Leibniz Rule” for material derivatives recorded in Lemma 6.7. This two lemmas allow us to compute the material derivatives of the linear momentum density $\rho u$ and of the angular momentum density $J \omega$. These computations come in very handy when deriving the local version of the balance of linear and angular momentum in Proposition 4.17.

**Lemma 4.15.** (Material derivative of the linear momentum density)

Let $(\Omega_0, \eta)$ be a continuum with Eulerian velocity $u$. If a conserved Eulerian mass measure is absolutely continuous with respect to the Lebesgue measure, then the following identity holds:
\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \rho (\partial_t u + (u \cdot \nabla) u).
\]

**Proof.** This follows from the local conservation of mass (which says that $D^\rho_t \rho = 0$) and Lemma 6.7:
\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = D^\rho_t (\rho u) = (D^\rho_t \rho) u + \rho (D^u_t u) = 0 + \rho (\partial_t u + (u \cdot \nabla) u).
\]

**Lemma 4.16.** (Material derivative of the angular momentum density)

Let $(\Omega_0, \eta, Q)$ be a micropolar continuum with Eulerian velocity $u$ and Eulerian angular velocity $\omega$. Let $\rho$ be an Eulerian mass density which is conserved, let $j$ be a conserved Eulerian microinertia density, and let $J := j \rho$ denote the microinertia. The following identity holds:
\[
\partial_t (J \omega) + \nabla \cdot (J \omega \otimes u) = J (\partial_t \omega + (u \cdot \nabla) \omega) + \omega \times J \omega.
\]

**Proof.** This follows from Corollary 4.10 and Lemma 6.7. Indeed, local conservation of microinertia says that $D^\rho_t J = [\Omega, J]$ where $\Omega = \text{ten} \, \omega$ is the Eulerian angular velocity tensor, and hence we obtain that, since $(\text{ten} \, \omega) \omega = \omega \times \omega = 0$,
\[
(D^\rho_t J) \omega = [\Omega, J] \omega = (\text{ten} \, \omega) J \omega - J (\text{ten} \, \omega) \omega = \omega \times (J \omega).
\]
So finally:
\[
\partial_t (J \omega) + \nabla \cdot ((J \omega) \otimes u) = D^\rho_t (J \omega) = (D^\rho_t J) \omega + J (D^u_t \omega) = \omega \times (J \omega) + J (\partial_t \omega + (u \cdot \nabla) \omega).
\]
We now come to the main result of this section, and one of the main results of this chapter, as we derive the local version of the balance of linear and angular momentum.

**Proposition 4.17.** *(Local conservation of linear and angular momentum)*

Let \( n = 3 \), let \((\Omega_0, \eta, Q, \nu, j)\) be a physical micropolar continuum governed by \((T, M)\) subject to \((f, \tau)\), and let

- \( u \) denote the Eulerian velocity,
- \( \omega \) denote the Eulerian angular velocity,
- \( \rho \) denote the Eulerian mass density, i.e. \( d\nu = \rho dL^3 \), and
- \( J = \rho j \) denote the Eulerian microinertia.

Then

\[
\rho \left( \partial_t u + (u \cdot \nabla) u \right) = \nabla \cdot T + f \quad \text{and} \quad J \left( \partial_t \omega + (u \cdot \nabla) \omega \right) + \omega \times (J \omega) = 2 \text{vec} T + \nabla \cdot M + \tau.
\]

**Proof.** The key idea is to use the Reynolds and divergence theorems (c.f. Theorem 2.7) to write all terms of the balance of linear and angular momentum as integrals over \( \mathcal{U}(t) \), no longer differentiated in time, and to then use the conservation of mass and the conservation of micro-inertia to simplify the resulting equations.

First we deal with linear momentum. The balance of linear momentum is

\[
\frac{d}{dt} \int_{\mathcal{U}(t)} \rho u = \int_{\partial \mathcal{U}(t)} T \cdot n + \int_{\mathcal{U}(t)} f.
\]

By the Reynolds transport theorem

\[
\frac{d}{dt} \int_{\mathcal{U}(t)} \rho u = \int_{\mathcal{U}(t)} \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u)
\]

and by the divergence theorem

\[
\int_{\partial \mathcal{U}(t)} T \cdot n = \int_{\mathcal{U}(t)} \nabla \cdot T.
\]

We may thus write the balance of linear momentum as

\[
\int_{\mathcal{U}(t)} \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \int_{\mathcal{U}(t)} \nabla \cdot T + f.
\]

Since \( \mathcal{U}_0 \subset \Omega_0 \) is arbitrary and \( \eta_t \) is a diffeomorphism it follows that the integral equation immediately above holds for arbitrary subsets \( \mathcal{U}(t) \subseteq \Omega(t) \) and hence the following PDE holds pointwise:

\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \nabla \cdot T + f \tag{4.1}
\]

In particular, note that local conservation of mass, and more specifically Lemma 4.15, tells us that

\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \rho (\partial_t u + (u \cdot \nabla) u).
\]

So finally:

\[
\rho \left( \partial_t u + (u \cdot \nabla) u \right) = \nabla \cdot T + f. \tag{4.2}
\]

Now we deal with angular momentum. We begin by recording some preliminary computations:

\[
\nabla \cdot ((x \times \rho u) \otimes u) = x \times \nabla \cdot (\rho u \otimes u), \tag{4.3}
\]

\[
x \times (Tn) = (x \times T) n, \quad \text{and} \tag{4.4}
\]

\[
\nabla \cdot (x \times T) = x \times (\nabla \cdot T) + 2 \text{vec} T, \tag{4.5}
\]

where we define \((v \times v)_{ij} := \epsilon_{iab} v_a T_{bj}\). Note that, in particular, it follows from (4.3) that

\[
\nabla^a_i (x \times \rho u) = x \times \nabla^a_i (\rho u).
\]

Indeed, (4.3), (4.4), and (4.5) follow from direct computations:

\[
(\nabla \cdot ((x \times \rho u) \otimes u))_i = \partial_j (\epsilon_{ikl} x_k \rho u_l u_j) = \epsilon_{ikl} x_k \partial_j (\rho u_l u_j) + \epsilon_{ikl} \delta_{jk} \rho u_l u_j
\]

\[
= (x \times \nabla \cdot (\rho u \otimes u))_i + \epsilon_{ijl} \rho u_l u_j.
\]
whilst using (4.4), the divergence theorem, and (4.5) yields
\[(\nabla \cdot (x \times T))_i = \partial_j (\epsilon_{ijkl} x_j T_{kl} n_l) = (x \times T)_{il} n_l = ((x \times T) n)_i,\]
and finally
\[(\nabla \cdot (x \times T))_i = \partial_j (\epsilon_{ijkl} x_k T_{lj} + \epsilon_{ikl} \delta_{lj} T_{ij}) = (x \times (\nabla \cdot T))_i + (2 \text{vec } T)_i.\]

Now recall that the balance of angular momentum is
\[
\frac{d}{dt} \left( \int_{U(t)} x \times \rho u + \int_{\partial U(t)} J \omega \right) = \int_{\partial U(t)} x \times (Tn) + \int_{\partial U(t)} Mn + \int_{U(t)} x \times f + \int_{U(t)} \tau.
\]

By the Reynolds transport theorem, (4.6), Lemma 4.15, and Lemma 4.16 we obtain that
\[
\frac{d}{dt} \left( \int_{U(t)} x \times \rho u + J \omega \right) = \int_{U(t)} D_{\tau}^u (x \times \rho u) + \int_{U(t)} D_{\tau}^\omega (J \omega)
\]
\[
= \int_{U(t)} x \times D_{\tau}^u (\rho u) + D_{\tau}^\omega (J \omega) = \int_{U(t)} x \times (\rho D_{\tau}^u u) + J (D_{\tau}^\omega \omega) + \omega \times (J \omega)
\]
whilst using (4.4), the divergence theorem, and (4.5) yields
\[
\int_{\partial U(t)} x \times (Tn) + Mn = \int_{\partial U(t)} (x \times T + M) n = \int_{U(t)} \nabla \cdot (x \times T + M)
\]
\[
= \int_{U(t)} x \times (\nabla \cdot T) + 2 \text{vec } T + \nabla \cdot M.
\]

We may thus write the balance of angular momentum as
\[
\int_{U(t)} x \times (\rho D_{\tau}^u u) + J (D_{\tau}^\omega \omega) + \omega \times (J \omega) = \int_{U(t)} x \times (\nabla \cdot T + f) + 2 \text{vec } T + \nabla \cdot M + \tau.
\]

Since \(U_0 \subseteq \Omega_0\) is arbitrary and \(\eta_t\) is a diffeomorphism it follows that the integral equation immediately above holds for arbitrary subsets \(U(t) \subseteq \Omega(t)\) and hence the following PDE holds pointwise:
\[
x \times (\rho D_{\tau}^u u) + J (D_{\tau}^\omega \omega) + \omega \times (J \omega) = x \times (\nabla \cdot T + f) + 2 \text{vec } T + \nabla \cdot M + \tau.
\]

In particular, since (4.2) says that
\[
\rho D_{\tau}^u u = \nabla \cdot T + f
\]
it follows that
\[
J (D_{\tau}^\omega \omega) + \omega \times (J \omega) = 2 \text{vec } T + \nabla \cdot M + \tau
\]
or, in more expansive but standard notation,
\[
J (\partial_t \omega + (u \cdot \nabla) \omega) + \omega \times (J \omega) = 2 \text{vec } T + \nabla \cdot M + \tau.
\]

\[\square\]

4.4. Energy. Having introduced the conservation of mass, microinertia, and linear and angular momentum, we conclude Section 4 by considering another conserved quantity: energy. We proceed with the process: we define the kinetic energy associated with a continuum, we compute its expression for a rigid body and use that to define an appropriate notion of the conservation of energy for micropolar continua, and finally we derive a local version of the balance of energy.

**Definition 4.18.** (Kinetic energy of a flow map)

Let \((\Omega_0, \eta)\) be a continuum with Eulerian velocity \(u\) and conserved mass measure \(\nu\). For any Borel subset \(E \subseteq \Omega(t)\) we define the kinetic energy of \(E\) to be \(K(E) := \frac{1}{2} \|u\|^2_{L^2(E)}\), i.e.

\[
K(E) = \frac{1}{2} \int_E |u(x,t)|^2 d\nu(x).
\]

As alluded to earlier, now that we have defined the kinetic energy associated with a continuum we compute its expression for a rigid body. This makes use of the decomposition of the Eulerian velocity of a rigid body established in Proposition 3.3.
Proposition 4.19. (Kinetic energy of 3-dimensional rigid motions)
Let \( n = 3 \), let \((\Omega_0, \eta)\) be a rigid body, and let \( \bar{u} \) and \( \bar{\omega} \) be as in Proposition 3.3. Then the kinetic energy of the rigid body is a quadratic form

\[
K : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)
\]

\[
(\bar{u}, \bar{\omega}) \mapsto \frac{1}{2} M|\bar{u}|^2 + \frac{1}{2} J : (\bar{\omega} \otimes \bar{\omega})
\]

where \( M \) and \( J \) are the mass and moment of inertia of the rigid body, respectively.

**Proof.** First we show that \( \bar{u} \) and \( \bar{\omega} \times (\cdot - \bar{x}) \) are orthogonal in \( L_2^\omega \). This is a direct computation:

\[
\left( \bar{u}, \bar{\omega} \times (\cdot - \bar{x}) \right)_{L_2^\omega} = M \int_{\Omega(t)} \bar{u} \cdot \bar{\omega} \times (x - \bar{x}) \, d\nu_t = M \bar{u} \cdot \bar{\omega} \times \left( \int_{\Omega(t)} x \, d\nu_t - \bar{x} \right) = 0
\]

where \( M := M(\mathcal{U}(t)) \). Since \( \bar{u} \) and \( \bar{\omega} \times (\cdot - \bar{x}) \) are orthogonal in \( L_2^\omega \), we know that

\[
K(\bar{u}, \bar{\omega}) = \frac{1}{2} \tilde{I} : (\bar{\omega} \otimes \bar{\omega}) + \frac{1}{2} \tilde{J} : (\bar{\omega} \otimes \bar{\omega})
\]

for some \( \tilde{I}, \tilde{J} \in \mathbb{R}^{3 \times 3} \). Let us compute \( \tilde{I} \) and \( \tilde{J} \):

\[
K(\bar{u}, \bar{\omega}) = \frac{1}{2} \|u\|^2_{L_2^\omega} = \frac{1}{2} \|ar{u}\|^2_{L_2^\omega} + \frac{1}{2} \|\bar{\omega} \times (\cdot - \bar{x})\|^2_{L_2^\omega}
\]

where

\[
\|\bar{u}\|^2_{L_2^\omega} = \int_{\Omega(t)} |\bar{u}|^2 \, d\nu_t = M|\bar{u}|^2
\]

i.e. \( \tilde{I} = MI \), and where

\[
\|\bar{\omega} \times (\cdot - \bar{x})\|^2_{L_2^\omega} = \int_{\Omega(t)} |\bar{\omega} \times (x - \bar{x})|^2 \, d\nu_t = \int_{\Omega(t)} \left( |\bar{\omega}|^2 |x - \bar{x}|^2 - |\bar{\omega} \cdot (x - \bar{x})|^2 \right) \, d\nu_t (x)
\]

\[
= \int_{\Omega(t)} |x - \bar{x}|^2 I - (x - \bar{x}) \otimes (x - \bar{x}) \, d\nu_t (x) : (\bar{\omega} \otimes \bar{\omega}) = M ((\text{tr} V) I - V) : (\bar{\omega} \otimes \bar{\omega}) = J : (\bar{\omega} \otimes \bar{\omega})
\]

where we have used Proposition 6.3 to expand \( |\bar{\omega} \times (x - \bar{x})|^2 \). So indeed \( \tilde{J} = J \). \( \square \)

Note that Proposition 4.19 helps motivate the definition of the kinetic energy density in Definition 4.20 below where we define what it means for energy to be conserved for a micropolar continuum.

**Definition 4.20.** (Purely mechanical micropolar continuum, balance of energy, and related notions)
Let \( n = 3 \). We say that a physical micropolar continuum \((\Omega_0, \eta, Q, \nu, J)\) governed by \((T, M)\) subject to \((f, \tau)\), where

- \( u \) denote the Eulerian velocity,
- \( \omega \) denote the Eulerian angular velocity,
- \( \rho \) denote the Eulerian mass density, i.e. \( d\nu = \rho d\mathcal{L}^3 \),
- \( J = \rho j \) denote the Eulerian microinertia,

is purely mechanical if there exist Eulerian functions defined along the flow with codomain \( \mathbb{R}_+ \), denoted \( \epsilon \) and \( \delta \), called respectively the mechanical energy density and the thermodynamic dissipation density, such that

\[
\frac{d}{dt} \left( \int_{\Omega(t)} (\epsilon + K) \right) + \int_{\Omega(t)} \delta = \int_{\Omega(t)} (T n) \cdot u + (M n) \cdot \omega + \int_{\Omega(t)} f \cdot u + \tau \cdot \omega
\]

where \( n \) is the outer unit normal and where \( K := \frac{1}{2} \rho |u|^2 + \frac{1}{2} J : \omega \otimes \omega \) is called the kinetic energy density. Moreover, the integral equation above is referred to as the balance of energy.

We are now ready to conclude this section by obtaining the local version of the balance of energy introduce in Definition 4.20 above.

**Theorem 4.21.** (Local conservation of energy)
Let \( n = 3 \), let \((\Omega_0, \eta, Q, \nu, J)\) be a purely mechanical physical micropolar continuum governed by \((T, M)\) subject to \((f, \tau)\), and let
• $u$ denote the Eulerian velocity,
• $\omega$ denote the Eulerian angular velocity,
• $\Omega$ denote the Eulerian angular velocity tensor,
• $\epsilon$ denote the mechanical energy density.

Then

$$\partial_t \epsilon + \nabla \cdot (\epsilon u) \leq (\nabla u - \Omega) : T + \nabla \omega : M.$$  

**Proof.** We begin by observing that, since the thermodynamic dissipation density $\delta$ is non-negative, we can re-write the balance of energy as the inequality

$$\frac{d}{dt} \left( \int_{U(t)} (\epsilon + K) \right) \leq \int_{\partial U(t)} (T \cdot n) \cdot u + (M \cdot n) \cdot \omega + \int_{U(t)} f \cdot u + \tau \cdot \omega,$$

which will hereafter be referred to as the energy inequality.

We now seek to rewrite all integrals appearing in the energy inequality in terms of non-time-differentiated integrals over the bulk domain $U(t)$. In particular, we use the Reynolds transport theorem to see that

$$\frac{d}{dt} \left( \int_{U(t)} (\epsilon + K) \right) = \int_{U(t)} \partial_t (\epsilon + K) + \nabla \cdot ((\epsilon + K) u)$$

and we use the divergence theorem to see that

$$\int_{\partial U(t)} (T \cdot n) \cdot u + (M \cdot n) \cdot \omega = \int_{U(t)} \nabla \cdot (u \cdot T) + \nabla \cdot (\omega \cdot M)$$

since indeed, for any differentiable matrix and vector fields $A$ and $v$ and any sufficiently regular open set $U$,

$$\int_{\partial U} (An) \cdot v = \int_{\partial U} A_{ij} n_j v_i = \int_{U} \partial_j (A_{ij} v_i) = \int_{U} \nabla \cdot (v \cdot A).$$

For simplicity, we now define the power density $P := \nabla \cdot (u \cdot T) + \nabla \cdot (\omega \cdot M) + f \cdot u + \tau \cdot \omega$. We can thus write the energy inequality as

$$\int_{U(t)} \partial_t (\epsilon + K) + \nabla \cdot ((\epsilon + K) u) \leq \int_{U(t)} \nabla \cdot (u \cdot T) + \nabla \cdot (\omega \cdot M) + f \cdot u + \tau \cdot \omega = \int_{U(t)} P$$

i.e.

$$\int_{U(t)} \partial_t \epsilon + \nabla \cdot (\epsilon u) \leq \int_{U(t)} P - (\partial_t K + \nabla \cdot (K u)).$$

Since $U_0 \subseteq \Omega_0$ is arbitrary and $\eta$ is a diffeomorphism it follows that the integral equation immediately above holds for arbitrary subsets $U(t) \subseteq \Omega(t)$ and hence the following differential inequality holds pointwise:

$$\partial_t \epsilon + \nabla \cdot (\epsilon u) \leq P - (\partial_t K + \nabla \cdot (K u)).$$

To conclude the proof, we simply compute $P - (\partial_t K + \nabla \cdot (K u)) = P - D_i^n K$. The key observation is that

$$D_i^n K = (\rho D_i^n u) \cdot u + (J D_i^n \omega + \omega \times J \omega) \cdot \omega. \quad (4.7)$$

Indeed, this follows from writing $K = ( \frac{1}{2} \rho u ) \cdot u + (J \omega) \cdot \omega$, using Lemma 6.7 and simplifying the result using Lemma 4.15, Lemma 4.16, and the symmetry of $J$:

$$D_i^n \left( \frac{1}{2} \rho u \cdot u \right) = D_i^n \left( \frac{1}{2} \rho u \right) \cdot u + \frac{1}{2} \rho u \cdot (D_i^n u) = \frac{1}{2} \rho (D_i^n u) \cdot u + \frac{1}{2} \rho u \cdot (D_i^n u) = (\rho D_i^n u) \cdot u$$

and

$$D_i^n \left( \frac{1}{2} J \omega \cdot \omega \right) = D_i^n \left( \frac{1}{2} J \omega \right) \cdot \omega + \frac{1}{2} J \omega \cdot D_i^n \omega = \left( \frac{1}{2} J (D_i^n \omega) + \frac{1}{2} \omega \times J \omega \right) \cdot \omega + \frac{1}{2} J \omega \cdot D_i^n \omega$$

$$= \frac{1}{2} \left( (J + J^T) (D_i^n \omega) + \omega \times J \omega \right) \cdot \omega = J D_i^n \omega \cdot \omega.$$
So finally, using (4.7), Proposition 4.17, Lemma 6.8, Proposition 6.13, and Lemma 6.17 we obtain that
\[ P - (\partial_t K - \nabla \cdot (K u)) = \nabla \cdot (u \cdot T + \omega \cdot M) + f \cdot u + \tau \cdot M - D^u K \]
\[ = \nabla u : T + \nabla \omega : M + (\nabla \cdot T + f) \cdot u + (\nabla \cdot M + \tau) \cdot \omega \]
\[ - (\rho D^u u) \cdot u - (J D^u \omega + \omega \times J \omega) \cdot \omega \]
\[ = \nabla u : T + \nabla \omega : M - (2 \text{vec} T) \cdot \omega \]
\[ = (\nabla u - \Omega) : T + \nabla \omega : M \]
such that indeed
\[ \partial_t u + \nabla \cdot (eu) \leq (\nabla u - \Omega) : T + \nabla \omega : M. \]

4.5. Boundary conditions. In this section we only briefly discuss boundary conditions associated with micropolar continua. The only purpose of the boundary conditions detailed here is to be such that the equations of motion for micropolar fluids ultimately derived are complete, and so we only consider so-called natural boundary conditions. That is not to say that there is not much to be done when it comes to discussing appropriate boundary conditions for micropolar continua in various contexts, but such a discussion is simply not the focus here.

This section follows the usual path: we define a version of the balance of linear and angular momentum taking into account boundary effects and derive the local version of this balance law.

**Definition 4.22.** (External boundary force and torque) Let \( n = 3 \), let \( (\Omega_0, \eta, Q, \nu, j) \) be a physical micropolar continuum governed by \( (T, M) \) subject to \( (f, \tau) \), and let
- \( u \) denote the Eulerian velocity,
- \( \omega \) denote the Eulerian angular velocity,
- \( \rho \) denote the Eulerian mass density, i.e. \( d\nu = \rho d\mathcal{L}^3 \), and
- \( J = \rho j \) denote the Eulerian microinertia.

Let \( f_b \) and \( \tau_b \) Eulerian functions defined along the flow with codomain \( \mathbb{R}^3 \) called the external boundary force and external boundary torque respectively. If, for every \( U_0 \subseteq \Omega_0 \),
\[ \frac{d}{dt} \int_{U(t)} \rho u \, dx = \int_{\partial U(t) \backslash \partial \Omega(t)} T n \, dx + \int_{\partial U(t) \cap \partial \Omega(t)} f_b \, dx + \int_{U(t)} f \, dx \]
and
\[ \frac{d}{dt} \left( \int_{U(t)} x \times \rho u \, dx + \int_{U(t)} J \omega \, dx \right) = \int_{\partial U(t) \backslash \partial \Omega(t)} x \times (T n) \, dx + \int_{\partial U(t) \cap \partial \Omega(t)} M n \, dx \]
\[ + \int_{\partial U(t) \cap \partial \Omega(t)} x \times f_b \, dx + \int_{\partial U(t) \cap \partial \Omega(t)} \tau_b \, dx + \int_{U(t)} x \times f \, dx + \int_{U(t)} \tau \, dx \]
where \( U(t) := \eta_t(U_0) \) for every \( t \geq 0 \), then we say that the physical micropolar continuum is subject to the boundary effects \( (f_b, \tau_b) \). Moreover, the two integral equations above are referred to as the balance of linear momentum for boundary flows and the balance of angular momentum for boundary flows respectively.

We now derive the local version of the balance of linear and angular momentum introduced above in **Definition 4.22**, which are called the natural boundary conditions.

**Proposition 4.23.** (Natural boundary conditions) Let \( n = 3 \), let \( (\Omega_0, \eta, Q, \nu, j) \) be a physical micropolar continuum governed by \( (T, M) \) subject to \( (f, \tau) \) and the boundary effects \( (f_b, \tau_b) \), and let
- \( u \) denote the Eulerian velocity,
- \( \omega \) denote the Eulerian angular velocity,
- \( \rho \) denote the Eulerian mass density, i.e. \( d\nu = \rho d\mathcal{L}^3 \), and
- \( J = \rho j \) denote the Eulerian microinertia.

Then
\[ T n = f_b \] and \[ M n = \tau_b \] on \( \partial \Omega(t) \),
where \( n \) denotes the outer unit normal to \( \Omega(t) \). These two equations are called the natural boundary conditions associated with the boundary effects \((f_b, \tau_b)\).

**Proof.** The key is to write
\[
\int_{\partial U(t) \setminus \partial \Omega(t)} = \int_{\partial U(t)} - \int_{\partial U(t) \cap \partial \Omega(t)}
\]
where the integrand is either \( Tn \) or \( x \times Tn + Mn \). Combining this with the balances of linear and angular momentum for boundary flows tells us that
\[
\frac{d}{dt} \int_{U(t)} \rho u - \int_{\partial U(t)} Tn - \int_{U(t)} f = \int_{\partial U(t) \cap \partial \Omega(t)} (f_b - Tn)
\]  
(4.8)
and
\[
\frac{d}{dt} \left( \int_{U(t)} x \times \rho u + \int_{U(t)} J\omega \right) - \int_{\partial U(t)} x \times Tn - \int_{\partial U(t)} Mn - \int_{U(t)} x \times f - \int_{U(t)} \tau = \int_{\partial U(t) \cap \partial \Omega(t)} x \times (f_b - Tn) + \int_{\partial U(t) \cap \partial \Omega(t)} (\tau_b - Mn).
\]  
(4.9)

Proceeding as in the proof of Proposition 4.17 tells us that the left-hand side of (4.8) is
\[
\int_{U(t)} \mathbb{D}_t^u (\rho u) - \nabla \cdot T - f
\]
which, by Proposition 4.17, vanishes. Similarly, the left-hand side of (4.9) may be written as
\[
\int_{U(t)} \mathbb{D}_t^u (x \times \rho u + J\omega) - \nabla \cdot (x \times J\omega) - \nabla \cdot M - x \times f - \tau,
\]
which also vanishes. So finally we deduce from (4.8) and the arbitrariness of \( U_0 \) that \( f_b = Tn \) on \( \partial \Omega(t) \). Plugging this into (4.9) and once again using the fact that \( U_0 \) is arbitrary tells us that \( \tau_b = Mn \) on \( \partial \Omega(t) \). \( \square \)

5. Constitutive equations

To begin this section let us comment on where we stand with respect to the derivation of the equations of motions for micropolar fluids. Combining Corollary 4.10 and Proposition 4.17 tells us what the equations of motion are. To close the system all that we have to do is specify how the stress tensor \( T \) and the couple stress tensor \( M \) depend on the dynamic variables \((u, \omega)\). This is precisely what we do in this section.

First we discuss frame-invariance in Section 5.1, then we record some results on the representation of frame-invariant linear maps in Section 5.2. We discuss the Onsager reciprocity relations in Section 5.3 and we conclude this section by putting it all together to derive the equations of motion of homogeneous incompressible Newtonian micropolar fluids in Section 5.4.

5.1. Frame-invariance. In this section we introduce the notions of frame-invariance for tensor-valued functions, and in particular for functions defined along the flow under the notion of similarity of continua and micropolar continua. We then compute how various kinematic quantities behave under similarity.

**Definition 5.1.** (Similar continua)
Given two continua \((\Omega_0, \eta)\) and \((\hat{\Omega}_0, \hat{\eta})\) we say that they are similar if there exists a time-dependent orientation-preserving rigid motion \( f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \) such that \( \Omega_0 = \hat{\Omega}_0 \) and \( \hat{\eta}_t = f_t \circ \eta_t \) for every \( t \geq 0 \). Moreover we say that \( f \) maps \( \eta \) to \( \hat{\eta} \).

Note that since rigid motions form a group under composition, similarity of continua is an equivalence relation on the set of continua. The notion of similar continua is important since constitutive equations, which postulate which quantities the stress tensor and couple stress tensor may depend on, must pass to the quotient induced by the equivalence relation of similarity. In other words: the constitutive equations for the stress and couple stress tensor must be well-defined on equivalence classes of continua. This is made precise in Definition 5.5 under the name of frame-invariance.

Of course, the similarity of continua can be interpreted physically: two continua \((\Omega_0, \eta)\) and \((\hat{\Omega}_0, \hat{\eta})\) are similar precisely when they correspond to the same system, but viewed by different observers.
Having defined the similarity of continua we now compute how various kinematic quantities behave under similarity.

**Proposition 5.2** (The behaviour of kinematic quantities of continua under similarity). Let \( (\Omega_0, \eta) \) and \( (\Omega_0, \tilde{\eta}) \) be similar continua where the rigid motion \( f \) which maps \( \eta \) to \( \tilde{\eta} \) is given by

\[
f(t, x) = b(t) + R(t)(x - b_0)
\]

for \( b_0 \in \mathbb{R}^n \), \( b : [0, \infty) \to \mathbb{R}^n \), and \( R : [0, \infty) \to SO(n) \). Then

1. \( \partial_t \tilde{\eta} = \dot{b} + \dot{R}(\eta - b_0) + \dot{R} \partial_t \eta \),
2. \( \nabla \tilde{\eta} = R \nabla \eta \),
3. \( \tilde{u} \circ f = \dot{b} + \dot{R}(\eta - b_0) + Ru \), and
4. \( (\nabla \tilde{u}) \circ f = \dot{R} R^T + R \nabla u R^T \),

where \( u \) and \( \tilde{u} \) denote the Eulerian velocities of \( (\Omega_0, \eta) \) and \( (\Omega_0, \tilde{\eta}) \) respectively.

**Proof.** Throughout this proof we will abuse notation to various degrees. We will omit the explicit dependence of the various functions on their variables, except when that dependence is essential to the computation being carried out. We will also abusively write \( f \circ \eta \) to denote \( f(t, \eta(t(x))) \) wherever it is helpful to do so for the brevity and clarity of the argument.

We begin by computing the derivatives of \( f \). It follows immediately from (5.1) that

\[
\partial_t f = \dot{b} + \dot{R}(\cdot - b_0) \quad \text{and} \quad \nabla f = R.
\]

We are now equipped to compute the derivatives of \( \tilde{\eta} \). Since \( \tilde{\eta}(t) = f(t, \eta(t)) \) it follows that

\[
\partial_t \tilde{\eta} = (\partial_t f) \circ \eta + ((\nabla f) \circ \eta) \partial_t \eta \quad \text{and} \quad \nabla \tilde{\eta} = ((\nabla f) \circ \eta) \nabla \eta
\]

such that, in light of (5.2),

\[
\partial_t \tilde{\eta} = \dot{b} + R(\eta - b_0) + \dot{R} \partial_t \eta \quad \text{and} \quad \nabla \tilde{\eta} = R \nabla \eta,
\]

i.e. (1) and (2) hold.

We now compute the Eulerian velocity \( \tilde{u} \). Since \( \tilde{u} \) is defined as \( \tilde{u} := \partial_t \tilde{\eta} \circ \tilde{\eta}^{-1} \) we must first compute the inverse of \( \tilde{\eta} \). Since \( \tilde{\eta} = f \circ \eta \) this is immediate:

\[
\tilde{\eta}^{-1} = \eta^{-1} \circ f^{-1}.
\]

Note that this really means that, for every \( t \geq 0 \),

\[
\tilde{\eta}^{-1}_t = \eta^{-1}_t \circ f^{-1}_t.
\]

We may now compute \( \tilde{u} \). Using (1) and (5.3) we see that

\[
\tilde{u} = \partial_t \tilde{\eta} \circ \tilde{\eta}^{-1} = \left( \dot{b} + R(\eta - b_0) + \dot{R} \partial_t \eta \right) \circ (\eta^{-1} \circ f^{-1}) = \left( \dot{b} + R(\cdot - b_0) + Ru \right) \circ f^{-1}
\]

from which (3) follows.

Finally we compute \( (\nabla \tilde{u}) \circ f \). We introduce \( (\nabla \tilde{u}) \circ f \) by differentiating both sides of (3), which yields

\[
((\nabla \tilde{u}) \circ f) \nabla f = \dot{R} + R \nabla u.
\]

Using (5.2) we may apply \( R^T \) to both sides of (5.4) and obtain that

\[
(\nabla \tilde{u}) \circ f = \dot{R} R^T + R \nabla u R^T,
\]

i.e. (4) holds. \( \square \)

We now define an analog of Definition 5.1 for the micropolar realm by defining the similarity of micropolar continua.

**Definition 5.3.** (Similar micropolar continua)

Given two micropolar continua \( (\Omega_0, \eta, Q) \) and \( (\Omega_0, \tilde{\eta}, \tilde{Q}) \) we say that they are similar if there exists a time-dependent orientation-preserving rigid motion \( f = b + R(\cdot - b_0) \) such that \( \tilde{\Omega}_0 = \Omega_0 \), \( \tilde{\eta}_t = f_t \circ \eta_t \), and \( \tilde{Q}_t = R_t Q_t \) for every \( t \geq 0 \). Moreover we say that \( f \) maps \( (\eta, Q) \) to \( (\tilde{\eta}, \tilde{Q}) \).

A justification for why we define the similarity of micropolar continua in this way is provided in Figure 4. Having defined similarity for micropolar continua we now proceed as we did for classical continua and compute how various kinematic quantities behave under similarity.
**Figure 4.** Two micropolar continua \((\Omega_0, \eta, Q)\) and \((\Omega_0, \tilde{\eta}, \tilde{Q})\) are similar when there exists a rigid motion \(f = b + R(\cdot - b_0)\) such that this diagram commutes, i.e. \(\tilde{\eta} = f \circ \eta\) and \(\tilde{Q} = RQ\).

**Proposition 5.4.** *(The behaviour of kinematic quantities of micropolar continua under similarity)*

Let \((\Omega_0, \eta, Q)\) and \((\Omega_0, \tilde{\eta}, \tilde{Q})\) be similar micropolar continua where \(f = b + R(\cdot - b_0)\) maps \((\eta, Q)\) to \((\tilde{\eta}, \tilde{Q})\). Then

1. Items (1)–(4) of Proposition 5.2 hold,
2. \(\partial_t \tilde{Q} = \dot{R}Q + R\partial_t Q\),
3. \(\tilde{\Omega} \circ f = \dot{R}R^T + R\Omega R^T\),
4. \(\tilde{\omega} \circ f = \text{vec} (\dot{R}R^T) + R\omega\), and
5. \((\nabla \tilde{\omega}) \circ f = R\nabla \omega R^T\).

**Proof.** We know that items (1)–(4) of Proposition 5.2 hold since the continua underlying similar micropolar continua must themselves be similar, so here \((\Omega_0, \eta)\) and \((\Omega_0, \tilde{\eta})\) are similar continua. Note that (2) is immediate since \(\tilde{Q} = RQ\). To obtain (3) we compute that

\[
\tilde{\Omega} = ((\partial_t \tilde{Q} Q^T) \circ \tilde{\eta}^{-1} = ((\dot{R}Q + R\partial_t Q)(Q^T R^T)) \circ (\eta^{-1} \circ f^{-1}) = (\dot{R}R^T + R(\partial_t Q)Q^T R^T) \circ (\eta^{-1} \circ f^{-1}) = (\dot{R}R^T + R\Omega R^T) \circ f^{-1}
\]
from which (3) follows. Since \( f \) is orientation-preserving we know that \( \det R = 1 \), and hence we may use Lemma 6.14 and apply \( \text{vec} \) to both sides of the chain of equalities immediately above to deduce (4). To derive (5) we proceed as we did to obtain item (4) of Proposition 5.2: we differentiate (4), which tells us that
\[
((\nabla \omega) \circ f) \nabla f = R \nabla \omega
\]
and use the fact that \( \nabla f = R \) to obtain (5).

We conclude this section with the definition of the notion of frame-invariance.

**Definition 5.5.** (Frame-invariance)

(1) (Vector functions) We say that \( F : \mathbb{R}^n \to \mathbb{R}^n \) is frame-invariant if \( F(Rv) = RF(v) \) for every \( v \in \mathbb{R}^n \) and every \( R \in SO(n) \).

(2) (Tensors functions) Let \( X, Y \subseteq \mathbb{R}^{n \times n} \) be closed under conjugation by orientation-preserving orthogonal matrices. We say that \( F : X \to Y \) is frame-invariant if \( F(RMR^T) = RF(M)R^T \) for every \( M \in X \) and every \( R \in SO(n) \).

(3) (Functions defined along the flow) Let \( \mathcal{T} \) be a function which maps continua to \( k \)-tensor fields along that flow. We say that \( \mathcal{T} \) is frame-invariant if for every similar continua \((\Omega, \eta)\) and \((\tilde{\Omega}, \tilde{\eta})\), where \( f \) which maps \( \eta \) to \( \tilde{\eta} \) is given by
\[
f(t, x) = b(t) + R(t)(x - b_0)\]
for some \( b_0 \in \mathbb{R}^n \), \( b : [0, \infty) \to \mathbb{R}^n \), and \( R : [0, \infty) \to \mathbb{R}^n \), if we write
\[
T := \mathcal{T}(\Omega, \eta) \quad \text{and} \quad \tilde{T} := \mathcal{T}(\tilde{\Omega}, \tilde{\eta})
\]
then
\[
\tilde{T}_{j_1, \ldots, j_k} \circ f = R_{j_1 i_1}R_{j_2 i_2} \cdots R_{j_k i_k}T_{i_1, \ldots, i_k}.
\]

Note that this definition applies *mutatis-mutandis* to functions mapping micropolar continua to \( k \)-tensor fields along that flow.

With the notion of frame-invariance in hand, we can look back at our computations from Proposition 5.2 and Proposition 5.4 to identify the kinematic quantities which are frame-invariant. We record this below.

**Corollary 5.6.** (Identification of the frame-invariant kinematic quantities)

Let \((\Omega, \eta, Q)\) be a micropolar continuum, let \( u \) denotes its Eulerian velocity, and let \( \omega \) be its Eulerian angular velocity. Then \( \nabla u - \omega \) and \( \nabla \omega \) are frame-invariant.

**Proof.** The frame-invariance of \( \nabla \omega \) is precisely item (5) of Proposition 5.4. The frame-invariance of \( \nabla u - \Omega \) follows from item (4) of Proposition 5.2 and item (3) of Proposition 5.4 since upon subtracting the former from the latter we see that
\[
\nabla \tilde{u} - \tilde{\Omega} = (\dot{R}R^T + R\nabla u R^T) - (\dot{R}R^T + R\Omega R^T) = R(\nabla u - \Omega)R^T.
\]

\[\square\]

### 5.2. Representation of frame-invariant linear maps.

In this section we record several results on the representation of frame-invariant linear maps, which are inspired by analogous results in [Gur81, Wan70a, Wan70b, Smi71].

We make the technical assumption that the dimension \( n \) be odd, which is not concerning for our purposes here since we ultimately wish to consider the case \( n = 3 \). Nonetheless, it seems that the results below should hold in arbitrary dimensions. The source of this technical restriction lies in the fact that we consider the angular velocity and angular momentum to be in \( \mathbb{R}^3 \) (and not in \( \text{Skew}(3) \)). Crucially: the identification of \( \mathbb{R}^3 \) and \( \text{Skew}(3) \) is made via \text{ten} and \text{vec}, which are not invariant under actions by \( O(3) \), but only invariant under actions by \( SO(3) \). A key tool in obtaining the representation formulae below is to consider, for an appropriately chosen unit vector \( v \), the transformation \( 2v \otimes v - I \). This matrix is always in \( O(n) \) but is only in \( SO(n) \) when \( n \) is odd, and this is precisely the source of our technical restriction.

We begin with a couple of lemmas that will come in handy when discussing the frame-invariance of linear maps whose domain lies within the space of symmetric matrices. First we note relate the commutativity of symmetric matrices to the invariance of eigenspaces.
LEMMA 5.7. (Commuting symmetric operators keep eigenspaces invariant)
Let $S$ and $T$ be real symmetric matrices. $S$ and $T$ commute if and only if the eigenspaces of $S$ are invariant under $T$.

PROOF. Suppose first that $S$ and $T$ commute and let $V \subseteq \mathbb{R}^n$ be an eigenspace of $S$ with eigenvalue $\lambda$. Then, for every $x \in V$, $STx = TSx = \lambda Tx$ such that $Tx \in V$ (since $Tx$ is an eigenvector of $S$ with eigenvalue $\lambda$) and so indeed the eigenspaces of $S$ are invariant under $T$.

Suppose now that the eigenspaces of $S$ are invariant under $T$ and let us write $S = \sum_i \lambda_i v_i \otimes v_i$ (such a decomposition exists since $S$ is symmetric). By assumption we know that, for every $i$, $T v_i$ belongs to the eigenspace of $v_i$ such that $S(T v_i) = \lambda_i T v_i = T S v_i$ and thus $S$ and $T$ commute on each of the eigenspaces of $S$. Since the union of the eigenspaces of $S$ constitutes all of $\mathbb{R}^n$ we conclude that $S$ and $T$ commute. \qed

We now note that frame-invariant functions whose domain is the space of symmetric matrices preserve eigenspaces.

LEMMA 5.8. (Frame-invariant functions preserve eigenspaces)
Let $n$ be odd and let $F : \text{Sym}(n) \rightarrow \mathbb{R}^{n \times n}$ be frame-invariant. Then, for every $S \in \text{Sym}(n)$, eigenvectors of $S$ are eigenvectors of $F(S)$.

PROOF. Let $S \in \text{Sym}(n)$ and let $v \in \mathbb{R}^n$ be an eigenvector of $S$. Let us define $R := 2v \otimes v - I$, which is an orthogonal transformation since it is symmetric and satisfies

$$(2v \otimes v - I)^2 = 4|v|^2 - 4v \otimes v - I = I.$$ 

Moreover, $R$ is orientation preserving. Indeed: let us complete $v$ to a basis $\{v, w_1, \ldots, w_{n-1}\}$ of $\mathbb{R}^n$. Then $Rv = v$ and $Rw_i = -w_i$ such that every element of that basis is an eigenvector of $R$. Therefore, since $n$ is odd, $\det R = 1 \cdot (-1)^{n-1} = 1$.

Geometrically, we may describe $R$ as the reflection through the line spanned by $v$. Indeed, as already mentioned above when showing that $R$ is orientation-preserving:

$$Rv = v \text{ and } Rw = -w \text{ if } w \perp v.$$ \hspace{1cm} (5.5)

In particular: for any eigenvector $w$ of $S$ distinct from $v$ we have that $Rw = -w$, so the eigenspaces of $S$ are invariant under $R$. As a consequence, we deduce from Lemma 5.7 that $S$ and $R$ commute. Combining this with the frame-invariance of $F$ tells us that

$$RF(S)R^T = F(RSR^T) = F(S),$$

i.e. $F(S)$ and $R$ also commute. So finally:

$$R(F(S)v) = F(S)(Rv) = F(S)v$$

which, by (5.5), may only occur if $F(X)v \in \text{span}\{v\}$. So indeed we may conclude that $v$ is an eigenvector of $F(S)$. \qed

As an immediate consequence of Lemma 5.8 we obtain the following corollary.

COROLLARY 5.9. Let $F : \text{Sym}(n) \rightarrow \mathbb{R}^{n \times n}$ be frame-invariant. Then $\text{im } F \subseteq \text{Sym}(n)$.

PROOF. Let $S \in \text{Sym}(n)$ and let us write $S = \sum_i \lambda_i v_i \otimes v_i$ where the $v_i$’s form an orthonormal basis of $\mathbb{R}^n$. By Lemma 5.8 we know that there exists $\mu_i$’s such that $F(S)$ has eigenpairs $\{(\mu_i, v_i)\}$, and therefore:

$$F(S) = \sum_i \mu_i v_i \otimes v_i$$

such that indeed $F(S)$ is symmetric. \qed

We are now ready to begin to establish a slew of representation formulae for frame-invariant linear maps. We begin with vector-to-vector frame-invariant linear maps.

LEMMA 5.10. (Representation formula for linear frame-invariant maps from $\mathbb{R}^n$ to itself)
Let $n$ be odd and let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear and frame-invariant. Then $F$ is a scalar multiple of the identity.
orthogonal matrices. Moreover, $\det R$ is frame-invariant, because
\[ \det R = 1 \cdot (-1)^{n-1} = 1 \] since $n$ is odd.
Therefore $M$ commutes with $2x \otimes x - I$, and hence with $x \otimes x$, such that
\[ (Mx) \otimes x = M(x \otimes x) = (x \otimes x)M = x \otimes (M^T x). \]
Applying both sides of this equality to $x$ we see that, since $x$ is a unit vector,
\[ Mx = (M^T x \cdot x)x = (Mx \cdot x)x. \]
In particular, since $M$ commutes with all orientation-preserving orthogonal matrices we may pick $R \in SO(n)$ such that $x = Re_1$ and observe that then
\[ MRe_1 \cdot Re_1 = R^T RM e_1 = Me_1 \cdot e_1 \]
and hence $Mx = (Me_1 \cdot e_1)x$ for all unit vectors $x$. So indeed $M$ is a multiple of the identity since $M = (Me_1 \cdot e_1)I$. \hfill \Box

We now prove a representation formula for linear frame-invariant maps from $\text{Sym}(n)$ to itself.

**Lemma 5.11. (Representation formula for linear frame-invariant maps from $\text{Sym}(n)$ to itself)**

Let $n$ be odd and let $F : \text{Sym}(n) \rightarrow \text{Sym}(n)$ be linear and frame-invariant. Then there exist $c_1, c_2 \in \mathbb{R}$ such that
\[ F(S) = c_1 (\text{tr } S) I + c_2 S \text{ for every } S \in \text{Sym}(n). \]

**Proof.** First we will show that the claim holds on the set of rank-1 symmetric matrices with unit norm. Let $v \in \mathbb{R}^n$ be a unit vector and let us consider the symmetric matrix $v \otimes v$. Lemma 5.8 tells us that, since $F$ is frame-invariant, $F(v \otimes v)$ has two eigenspaces: span $\{v\}$ and its orthogonal complement. So either the eigenspaces of $F(v \otimes v)$ are the same as those of $v \otimes v$ or the sole eigenspace of $F(v \otimes v)$ is $\mathbb{R}^n$. Either way we have that
\[ F(v \otimes v) = \tilde{c}_1 (v)v \otimes v + \tilde{c}_2 (v)(I - v \otimes v) = c_1 (v)v \otimes v + c_2 (v)I \]
for some $c_1, c_2 : S^{n-1} \rightarrow \mathbb{R}$.

Now let $w \in \mathbb{R}^n$ be another unit vector and let $R$ be the orientation-preserving orthogonal transformation which takes $v$ to $w$, i.e. $Rv = w$. Then, by the frame-invariance of $F$,
\[
0 = F(R(v \otimes v)R^T) - F(w \otimes w) = RF(v \otimes v)R^T - F(w \otimes w) \\
= R((c_1 (v)v \otimes v + c_2 (v)I)R^T - (c_1 (w)v \otimes v + c_2 (w)w)I) \\
= (c_1 (v) - c_1 (w))w \otimes w + (c_2 (v) - c_2 (w))w.)I.
\]
So finally, since $w \otimes w$ and $I$ are linearly independent we know that $c_1$ and $c_2$ are constants on $S^{n-1}$, i.e.
\[ F(v \otimes w) = c_1 v \otimes w + c_2 I, \quad (5.6) \]
for every $v \in S^{n-1}$. We have thus just shown that the representation formula holds on the set of symmetric matrices of rank-1 with unit norm (up to switching $c_1$ and $c_2$).

To conclude we leverage the linearity of $F$. Let $S$ be an arbitrary real symmetric matrix and let us decompose it as
\[ S = \sum_i \lambda_i v_i \otimes v_i \]
where each of the $v_i$ has unit norm. Then, by (5.6) and the linearity of $F$ we deduce that
\[ F(S) = \sum_i \lambda_i (c_1 v_i \otimes v_i + c_2 I) = c_1 S + c_2 (\text{tr } S)I \]
which is precisely the representation formula we sought (up to interchanging $c_1$ and $c_2$). \hfill \Box
We now prove a result similar to Lemma 5.11 but considering frame-invariant linear maps whose domain now lies within the space of skew-symmetric matrices. Moreover we restrict our attention to dimension $n = 3$ in order to be able to use ten and vec and thus streamline the argument.

**Lemma 5.12.** *(Representation formula for linear frame-invariant maps from $\text{Skew}(3)$ to $\mathbb{R}^{3 \times 3}$)*

Let $n = 3$ and let $F : \text{Skew}(3) \to \mathbb{R}^{3 \times 3}$ be linear and frame-invariant. Then $F$ is a scalar multiple of the identity.

**Proof.** Let $A \in \text{Skew}(3)$ with $|A|^2 = A : A = 2$ and let $v \in \mathbb{R}^3$ be defined as $v := \text{vec} A$. Note that we have chosen the seemingly odd normalization $|A|^2 = 2$ since it ensures that $v$ is a unit vector. Indeed, by Lemma 6.17,

$$|v|^2 = \text{vec} A : \text{vec} A = \frac{1}{2} A : \text{ten} \text{vec} A = \frac{1}{2} |A|^2 = 1.$$ 

It will be helpful throughout to recall the geometric interpretation of the action of $A$: $A$ is a (counter-clockwise) rotation by $\frac{\pi}{2}$ in the plane orthogonal to $v$ and annihilates vectors colinear with $v$.

Before beginning the proof in earnest we record some useful computations about $A$ whose outcomes are obvious in light of the geometric interpretation of the action of $A$. First, note that

$$A(v \otimes v) = (v \otimes v)A = 0. \quad (5.7)$$

Indeed, for any $w \in \mathbb{R}^3$,

$$A(v \otimes w)w = (v \cdot w)Aw = (v \cdot w)(v \times w) = 0 \quad \text{and} \quad (v \otimes v)Aw = (v \cdot Aw)v = (v \cdot (v \times w))v = 0.$$ 

Second, observe that

$$-A^2 = I - v \otimes v = \text{proj}_{\{v\}^\perp}. \quad (5.8)$$

Indeed: $-A^2v = -A(v \times v) = 0$ whilst, for any $w \bot v$,

$$-A^2w = -(v \times (v \times w)) = -(v \cdot w)v + (v \cdot v)w = w.$$ 

**Step 1.** It will be very convenient for the remainder of this argument to consider the transformation $R := A + v \otimes v$. In particular we will show that $R$ is orthogonal, orientation-preserving, and commutes with $F(A)$. The orthogonality of $R$ follows from (5.7) and (5.8) since

$$RR^T = (A + v \otimes v)(-A + v \otimes v) = -A^2 + v \otimes v = I$$

and, similarly, $R^TR = I$. We can understand the action of $R$ geometrically by comparing it to the action of $A$: $R$ acts in the same way as $A$ on vectors orthogonal to $v$ but is the identity on the span of $v$. In particular, this tells us that $R$ can be decomposed as the direct sum of two orientation-preserving transformations, so $R$ itself is orientation-preserving. Now let us show that $R$ and $F(A)$ commute. Observe that, by (5.7) and (5.8),

$$RAR^T = (A + v \otimes v)A(-A + v \otimes v) = -A^3 = A(I - v \otimes v) = A$$

and hence, since $F$ is frame-invariant,

$$F(A) = F(RAR^T) = RF(A)R^T$$

such that indeed $F(A)$ and $R$ commute.

**Step 2.** We now show that the actions of $A$ and $F(A)$ agree on span $\{v\}$ and span $\{v\}^\perp$. To do so, observe that we may characterize the action of $R^2$ as follows:

$$\begin{cases} 
R^2x = x \iff x \in \text{span} \{v\} \quad \text{and} \\
R^2x = -x \iff x \bot v.
\end{cases} \quad (5.9)$$

Indeed:

$$R^2 - I = (A + v \otimes v)^2 - I = A^2 + v \otimes v - I = -(I + v \otimes v) + v \otimes v - I = -2(I - v \otimes v)$$

and thus $R^2 + I = 2v \otimes v = 2\text{proj}_{\text{span}\{v\}}$.

Since $R$ and $F(A)$ commute we may then compute that, for any $w \bot v$,

$$\begin{cases} 
R^2F(A)v = F(A)R^2v = F(A)v \quad \text{and} \\
R^2F(A)w = F(A)R^2w = -F(A)w.
\end{cases}$$
such that, by (5.9),
\[ F(A)v \in \text{span}\{v\} \text{ and } F(A)w \perp v. \quad (5.10) \]

Now let us pick \( w_1, w_2 \in \mathbb{R}^3 \) to be unit vectors orthogonal to \( v \) such that \( (v, w_1, w_2) \) forms an oriented orthonormal basis of \( \mathbb{R}^3 \), i.e.
\[ v \times w_1 = w_2, \quad w_1 \times w_2 = v, \quad \text{and } w_2 \times v = w_1. \]

Note that, since \( w_1, w_2 \perp v \), we have that
\[ Rw_1 = Av_1 = v \times w_1 = w_2 \quad \text{and} \quad Rw_2 = Av_2 = v \times w_2 = -w_1. \]

Therefore, since \( R \) and \( F(A) \) commute,
\[
RF(A)w_1 = F(A)Rw_1 = F(A)w_2 \quad \text{and} \quad RF(A)w_2 = F(A)Rw_2 = -F(A)w_1
\]
and hence, in light of (5.10), \((F(A)v, F(A)w_1, F(A)w_2)\) forms an oriented orthogonal basis of \( \mathbb{R}^3 \).

This allows us to characterize the behaviour of \( F(A) \) on \( \text{span}\{v\}^\perp \): \( F(A) \) takes the orthonormal basis \( \{w_1, w_2\} \) of \( \text{span}\{v\}^\perp \), which complements \( v \) to an oriented orthonormal basis \((v, w_1, w_2)\) of \( \mathbb{R}^3 \), to an orthogonal basis \((F(A)v, F(A)w_1, F(A)w_2)\) of \( \text{span}\{v\}^\perp \), which complements \( v \) to an oriented orthogonal basis \((F(A)v, F(A)w_1, F(A)w_2)\) of \( \mathbb{R}^3 \). So indeed the actions of \( F(A) \) and \( A \) agree on \( \text{span}\{v\}^\perp \). Combining this observation with (5.10) allows us to conclude that
\[ F(A) = c_1(A)v \otimes v + c_2(A)A \]
for some scalars \( c_1(A), c_2(A) \in \mathbb{R} \).

**Step 3.** We know show that \( F \) is constant on the set of skew-symmetric matrices of norm \( \sqrt{2} \). Let \( A \) and \( B \) be 3-by-3 skew-symmetric matrices with \( |A|^2 = |B|^2 = 2 \) and let \( v := \text{vec} A \) and \( w := \text{vec} B \). Then, by Step 2, there exist scalars \( c_1(A), c_2(A), c_1(B), \) and \( c_2(B) \in \mathbb{R} \) such that
\[ F(A) = c_1(A)v \otimes v + c_2(A)A \quad \text{and} \quad F(B) = c_1(B)w \otimes w + c_2(B)B. \]

Now let \( Q \in SO(3) \) such that \( w = Qv \), and hence, by Lemma 6.14, \( B = QAQ^T \). Then, by frame-invariance of \( A \) we have that
\[ c_1(B)w \otimes w + c_2(B)B = F(B) = QF(A)Q^T = Q(c_1(A)v \otimes v + c_2(A)A)Q^T = c_1(A)w \otimes w + c_2(A)B. \]

Since \( w \otimes w \) and \( B \) are linearly independent we deduce that \( c_1(A) = c_1(B) \) and \( c_2(A) = c_2(B) \), so indeed \( F \) is constant on \( \{ A \in \text{Skew}(3) : |A|^2 = 2 \} \).

**Step 4.** We conclude by linearity of \( F \). Since there exist scalars \( c_1, c_2 \in \mathbb{R} \) such that
\[ F(A) = c_1 \text{vec} A \otimes \text{vec} A + c_2 A \]
for every 3-by-3 skew-symmetric matrix \( A \) satisfying \( |A|^2 = 2 \) it follows that, for any \( B \in \text{Skew}(3) \),
\[ F(B) = \frac{|B|}{\sqrt{2}} F \left( \frac{\sqrt{2}}{|B|} B \right) = c_1 \sqrt{2} \text{ vec} B \otimes \text{vec} B + c_2 B. \]

In particular, leveraging once again the linearity of \( F \) we note \( F(B) - F(C) - F(B - C) \) must vanish for all \( B, C \in \text{Skew}(3) \), from which we deduce that \( c_1 = 0 \). So finally:
\[ F(B) = c_2 B \quad \text{for every } B \in \text{Skew}(3). \]

We conclude this section with a representation formulae for general frame-invariant linear maps from \( \mathbb{R}^{n \times n} \) to itself (i.e. we do not specify, as we did in the previous two results, that the domain is contained in either the space of symmetric matrices or the space of skew-symmetric matrices). Once again we restrict our attention to dimension \( n = 3 \).

**Proposition 5.13.** (Representation formula for linear frame-invariant maps from \( \mathbb{R}^{3 \times 3} \) to itself)
Let \( n = 3 \) and let \( F : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) be linear and frame-invariant. Then there exists \( c_1, c_2, c_3 \in \mathbb{R} \) such that
\[ F(M) = c_1(\text{tr} M)I + c_2 \text{Sym}(M) + c_3 \text{Skew}(M) \quad \text{for every } M \in \mathbb{R}^{n \times n}. \]
Proof. Equipped with Corollary 5.9, Lemma 5.11, Lemma 5.12, and the observation that Sym + Skew = id, the only details that remain to be checked are that Sym and Skew are frame-invariant. This follows immediately from the observation that, for any $M \in \mathbb{R}^{3 \times 3}$ and any $R \in SO(3)$, $(RMRT)^T = R(M^T)R^T$. Therefore
\[
\text{Sym}(RMRT) = \frac{1}{2}(RMRT + RMTR^T) = R\text{Sym}(M)R^T
\]
and
\[
\text{Skew}(RMRT) = \frac{1}{2}(RMRT - RMTR^T) = R\text{Skew}(M)R^T.
\]
Using the linearity of $F$ we may then decompose
\[
F = F \circ \text{Sym} + F \circ \text{Skew},
\]
where
- $F \circ \text{Sym} : \text{Sym}(3) \to \mathbb{R}^{3 \times 3}$ is linear and frame-invariant, and so by Corollary 5.9 and Lemma 5.11 above we know that $F \circ \text{Sym} = c_1 \text{tr}(\cdot) + c_2 \text{id}$ for some scalars $c_1, c_2 \in \mathbb{R}$ and
- $F \circ \text{Skew} : \text{Skew}(3) \to \mathbb{R}^{3 \times 3}$ is linear and frame-invariant, and so by Lemma 5.12 above we know that $F \circ \text{Skew} = c_3 \text{id}$ for some scalar $c_3 \in \mathbb{R}$.

Putting it all together we see that indeed
\[
F = F \circ \text{Sym} + F \circ \text{Skew} = c_1 \text{tr}(\cdot) + c_2 \text{Sym} + c_3 \text{Skew}.
\]
\[\square\]

5.3. Onsager reciprocity relations. The Onsager reciprocity relations were proposed by Onsager [Ons31a, Ons31b] to provide a theoretical justification for the fact that, in some physical systems, the irreversibility of an underlying microscopic process leads to symmetry properties of some macroscopic observables. This fact has since been extensively verified experimentally [Mil60]. As pointed out in [MRP16], the fact that the symmetry arises at the macroscopic level as a consequence of irreversibility at a microscopic level is a purely mathematical feature, as shown in Theorem 5.14 below. Note that the statement of Theorem 5.14 is taken from [MRP16] and its proof is taken from [dGM62].

It is also worth pointing out that the Onsager reciprocity relations need not be invoked when the equations of motion of classical fluids are derived. This is explained in more detail in Remark 5.19 below.

Theorem 5.14. (Onsager reciprocity relations)
Let $X_t$ be a Markov process in $\mathbb{R}^n$ with transition kernel $P_t(dx|x_0)$ and invariant measure $\mu(dx)$. Define the expectation $z_t(x_0)$ of $X_t$ given that $X_0 = x_0$, i.e.
\[
z_t(x_0) = \mathbb{E}_{x_0} X_t = \int_0^T P_t(dx|x_0).
\]
Assume that
(1) $\mu$ is reversible, i.e. $\mu(dx_0)P_t(dx|x_0) = \mu(dx)P_t(dx_0|x)$ for every $x, x_0 \in \mathbb{R}^n$ and every $t > 0$,
(2) $\mu$ is Gaussian with mean zero and covariance matrix $G$, and
(3) $t \mapsto z_t(x_0)$ satisfies the equation $\dot{z}_t = -Az_t$ for some nonnegative matrix $A$.

Then $M := AG$ is symmetric.

Proof. It follows immediately from item 3 that, for every $t > 0$ and every $x_0 \in \mathbb{R}^n$,
\[
z_t(x_0) = e^{-At} x_0.
\]
Taking the outer product of (5.11) with $x_0$ and integrating with respect to $\mu(dx_0)$ tells us that
\[
\int x_0 \otimes z_t(x_0) \mu(dx_0) = \int x_0 \otimes e^{-At} x_0 \mu(dx_0).
\]
By definition of $z$, reversibility of $\mu$, and (5.11) we note that we may rearrange the left-hand side to see that
\[
\int x_0 \otimes z_t(x_0) \mu(dx_0) = \int x_0 \otimes \left( \int xP_t(dx|x_0) \right) \mu(dx_0) = \int \int x_0 \otimes xP_t(dx|x_0) \mu(dx_0) \\
= \int \int x_0 \otimes xP_t(dx|x_0|x) \mu(dx) = \int \left( \int x_0 P_t(dx|x) \right) \otimes x \mu(dx) \\
= \int z_t(x) \otimes x \mu(dx) = \int e^{-At} x \otimes x \mu(dx).
\]

In particular, since $\mu$ is a centered Gaussian with covariance matrix $G$ we note that $\int y \otimes y \mu(dx) = G$ and hence
\[
\int e^{-At} x \otimes x \mu(dx) = e^{-At} G \quad \text{and} \quad \int x_0 \otimes e^{-At} x_0 \mu(dx_0) = \left( \int x_0 \otimes x_0 \mu(dx_0) \right) e^{-At} = Ge^{-At}
\]
such that
\[
e^{-At} G = Ge^{-At}
\]
holds for all $t > 0$. Differentiating in time and setting $t = 0$, we deduce that
\[AG = GA^T\]
such that indeed $M = AG$ is symmetric. \qed

5.4. Micropolar fluids: definition and derivation of their constitutive equations. This section is the conclusion of this chapter, where we define micropolar fluids and derive their equations of motion. First we define various classes of micropolar continua. In particular, note that we will only consider homogeneous incompressible continua in the sequel.

Definition 5.15. (Homogeneity, incompressibility, and isotropy of micropolar continua)
Let $(\Omega_0, \eta, Q, \nu, j)$ be a physical micropolar continuum. We say that it is
- **homogeneous** if $d\nu = \rho d\mathcal{L}^3$ for some constant $\rho > 0$,
- **incompressible** if the flow map $\eta$ is incompressible, and
- **isotropic** if $j = I$, i.e. if the microinertia density is constant and equal to the identity matrix.

We are now ready to define a micropolar fluid.

Definition 5.16. (Homogeneous incompressible micropolar fluid and homogeneous incompressible Newtonian micropolar fluid)
A homogeneous incompressible micropolar fluid is a homogeneous incompressible physical micropolar continuum governed by $(T, M)$ for which there exist a scalar function $\rho$ defined along the flow, called the pressure, as well as functions $\hat{T}, \hat{M} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ and $\hat{\varepsilon} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$, which are independent of the flow, such that
\[
T = -pI + \hat{T} (\nabla u, \omega, \nabla \omega), \quad M = \hat{M} (\nabla u, \omega, \nabla \omega), \quad \text{and} \quad \varepsilon = \hat{\varepsilon} (\nabla u, \omega, \nabla \omega)
\]
where $u$ denotes the Eulerian velocity, $\omega$ denotes the Eulerian angular velocity, and $\varepsilon$ denotes the mechanical energy density. Moreover, if both $\hat{T}$ and $\hat{M}$ are linear then the homogeneous incompressible micropolar fluid is said to be Newtonian.

Now that we have defined micropolar fluids we seek to use frame-invariance and the balance of energy to establish precisely in what ways the stress tensor and couple stress tensor depend on the dynamic variables and their gradients. First we note that, for incompressible micropolar fluids, the mechanical energy density must be constant.

Proposition 5.17. (Constancy of the mechanical energy in incompressible fluids)
Consider a homogeneous incompressible micropolar fluid governed by $(T, M)$. If the fluid is purely mechanical then its mechanical energy density $\varepsilon$ must be constant.

Proof. As proven in Theorem 4.21 and in light of the Eulerian velocity $u$ being divergence-free (by incompressibility), the inequality
\[
\partial_t \varepsilon + u \cdot \nabla \varepsilon - T : (\nabla u - \Omega) = M : \nabla \omega \leq 0
\]
holds. To make the computation more palatable, let us write \( \hat{\epsilon} = \epsilon(A, v, B) \). Then, since incompressibility and Lemma 6.7 tells us that \( D^\epsilon \) obeys the Leibniz Rule, we may compute that

\[
(\partial_A \hat{\epsilon})(\nabla u, \omega, \nabla \omega) : D^\epsilon_t \nabla u + (\partial_e \hat{\epsilon})(\nabla u, \omega, \nabla \omega) : D^\epsilon_t \nabla \omega 
= -T : (\nabla u - \nabla) - M : \nabla \omega \leq 0. \tag{5.12}
\]

Since the inequality (5.12) must hold for all possible flows, the key observation is that for any \( A, B, C, D \) and any vectors \( e, f \in \mathbb{R}^n \) we may construct a flow such that, at some space-time point \((t, x)\),

\[
D^\epsilon_t \nabla u = A, \quad \nabla u = B, \quad D^\epsilon_t \nabla \omega = C, \quad \nabla \omega = D, \quad D^\epsilon_t \omega = e, \quad \text{and} \quad D \omega = f.
\]

Therefore, since \( D^\epsilon_t \nabla u, D^\epsilon_t \omega, \) and \( D^\epsilon_t \nabla \omega \) appear linearly in (5.12), we may violate the inequality unless \( \partial_A \hat{\epsilon} = \partial_B \hat{\epsilon} = 0 \) and \( \partial_e \hat{\epsilon} = 0 \), in which case \( \epsilon \) is indeed constant. \( \square \)

It is worth noting that the constancy of the mechanical energy proven in Proposition 5.17 above is a feature of incompressibility. Indeed, this result is a mathematical manifestation of the physical observation that incompressible continua are incapable of exerting mechanical work.

Our goal remains to establish the dependence of the stress tensor and the couple stress tensor on the dynamic variables and their gradients. In particular, to do so we need to invoke the Onsager reciprocity relations, which are phrased in terms of the inequality obtained in Theorem 4.21. The corresponding postulate is stated below.

**Definition 5.18.** (Dissipation inequality and Onsager reciprocity relations)
Consider a homogeneous incompressible micropolar fluid governed by \((T, M)\) that is purely mechanical. The inequality

\[
T : (\nabla u - \Omega) + M : \nabla \omega \geq 0
\]

is called the dissipation inequality. In particular, for \( \mathcal{Y} := (\nabla u - \Omega, \nabla \omega) \) and \( \mathcal{J}(\mathcal{Y}) := (T, M) \) the dissipation inequality may be written as

\[
\mathcal{Y} \cdot \mathcal{J}(\mathcal{Y}) \geq 0.
\]

We say that the micropolar fluid obeys the Onsager reciprocity relations if \( \nabla \mathcal{J} \) is symmetric, i.e. \( \partial_i \mathcal{J}_j = \partial_j \mathcal{J}_i \) for every \( i, j \).

Note that in the context of continuum mechanics the dissipation inequality is also known as the Clausius-Duhem inequality.

**Remark 5.19.** Note that, when we derive the equations of motion of classical fluids (e.g. the Euler or Navier-Stokes) equations by arguments from rational continuum mechanics we do not appeal to the Onsager reciprocity relations since the symmetry is guaranteed by the dissipation inequality and frame-invariance.

Indeed, for classical fluids the Clausius-Duhem inequality reads

\[
T(\nabla u) : \nabla u = T(\nabla u) : \nabla u \geq 0.
\]

Under the Newtonian assumption which postulates that the stress tensor \( T \) is linear, we deduce from frame-invariance arguments (and Lemma 5.11 in particular) that \( T = c_1 \text{tr}(\cdot) I + c_2 \text{id} \) for some constants \( c_1, c_2 \). In particular \( T \) is necessarily symmetric.

We are now ready to state and prove the necessary form of the constitutive equations relating the stress tensor and the couple stress tensor to the dynamic variables and their gradients.

**Theorem 5.20.** (Constitutive equations of homogeneous incompressible Newtonian micropolar fluids)
Consider a homogeneous incompressible micropolar fluid governed by \((T, M)\) that is purely mechanical. If \( \hat{T} \) and \( \hat{M} \) are frame-invariant and the micropolar fluid satisfies Onsager’s reciprocity relations then there exist universal constants \( \mu, \kappa, \alpha, \beta, \gamma \geq 0 \) such that

\[
T = \mu \nabla u + \kappa \text{ten} \left( \frac{1}{2} \nabla \times (u - \omega) \right) - pI \quad \text{and} \quad M = \alpha (\nabla \cdot \omega) I + \beta \nabla \omega + \gamma \text{ten} \nabla \times \omega.
\]

**Proof.** In light of Corollary 5.6 the frame-invariance of \( \hat{T} = \hat{T}(\nabla u, \omega, \nabla \omega) \) and \( \hat{M} \) tells us that we may write

\[
\hat{T} = \hat{T}(\nabla u - \Omega, \nabla \omega) \quad \text{and} \quad \hat{M} = \hat{M}(\nabla u - \Omega, \nabla \omega).
\]
Since $\hat{T}$ and $\hat{M}$ are linear we may write $\hat{L} := (\hat{T}, \hat{M})$ for some linear operator $\hat{L}$ which, by Onsager’s reciprocity relations, is symmetric. This means that $\hat{L}$ is completely determined by the quadratic form it generates.

Adapting the arguments from Section 5.1 and Wang’s paper on the representation of frame-invariant (or, in the terminology he employs, isotropic) functions we deduce that, since $\hat{T}$ and $\hat{M}$ are linear and frame-invariant and since $u$ is divergence-free,

$$\hat{T}(\nabla u - \Omega, \nabla \omega) = c_1 \nabla u + c_2 (\text{Skew} \nabla u - \Omega) + c_3 (\nabla \cdot \omega) I + c_4 \nabla \omega + c_5 \text{Skew} \nabla \omega$$

and

$$\hat{M}(\nabla u - \Omega, \nabla \omega) = c_6 (\nabla \cdot u) I + c_7 \nabla u + c_8 (\text{Skew} \nabla u - \Omega) + c_9 (\nabla \cdot \omega) I + c_{10} \nabla \omega + c_{11} \text{Skew} \nabla \omega$$

for some constants $c_1, c_2, \ldots, c_{11} \in \mathbb{R}$.

The quadratic form generated by $\hat{L}$ is therefore

$$\hat{L}(\nabla u - \Omega, \nabla \omega) \cdot (\nabla u - \Omega, \nabla \omega) = \hat{T} : (\nabla u - \Omega) + \hat{M} : \nabla \omega = c_1 |\nabla u|^2 + c_2 |\text{Skew} \nabla u - \Omega|^2 + c_3 |\nabla \cdot \omega|^2 + c_4 |\nabla \omega|^2 + c_5 |\text{Skew} \nabla \omega|^2$$

and so we deduce that $c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$. In particular this tells us that $\hat{T} = \hat{T}(\nabla u - \Omega)$ and that $\hat{M} = \hat{M}(\nabla \omega)$.

Finally we seek to leverage the dissipation inequality

$$\hat{T}: (\nabla u - \Omega) + \hat{M}: \nabla \omega \geq 0$$

to obtain sign conditions on the coefficients appearing in $T$ and $M$. To make this process easier we group the terms in $\hat{T}$ and $\hat{M}$ according to the orthogonal decomposition (with respect to the Frobenius inner product)

$$\mathbb{R}^{n \times n} \cong \mathbb{R} I \oplus \text{Dev}(n) \oplus \text{Skew}(n).$$

Recalling the identities

$$\nabla v = \nabla^0 v + \frac{2}{3} (\nabla \cdot v) I \quad \text{and} \quad \text{Skew} \nabla v = \frac{1}{2} \text{ten} \nabla \times v$$

from Lemma 6.11 and Lemma 6.16 respectively, we write

$$\hat{T}(\nabla u - \Omega) = c_2 \nabla u + c_3 \text{ten} \left( \frac{1}{2} \nabla \times u - \omega \right)$$

and

$$\hat{M}(\nabla \omega) = (c_9 + (2/3)c_{10})(\nabla \cdot \omega) I + c_{10} \nabla^0 \omega + (c_{11}/2) \text{ten} \nabla \times \omega.$$  

Defining $\mu = c_2$, $\kappa = c_3$, $\alpha = c_9 + (2/3)c_{10}$, $\beta = c_{10}$, and $\gamma = c_{11}/2$ and employing Lemma 6.17, the dissipation inequality now read

$$\mu |\nabla u|^2 + 2\kappa [(1/2) \nabla \times u - \omega]^2 + \alpha |\nabla \cdot \omega|^2 + \beta |\nabla^0 \omega|^2 + 2\gamma |\nabla \times \omega|^2 \geq 0.$$  

(5.14)

This inequality must hold for arbitrary flows. Since we can construct flows with arbitrary values of $\nabla u - \Omega$ and $\nabla \omega$ we deduce from (5.14) and the orthogonality of the decomposition (5.13) that $\mu, \kappa, \alpha, \beta, \gamma \geq 0$. □

Finally, we may now state and prove the main result of this chapter which establishes the equations of motion for homogeneous incompressible Newtonian micropolar fluids.

**Corollary 5.21.** (Equations of motions for homogeneous incompressible Newtonian micropolar fluids) Let $(\mathbb{R}^3, \eta, Q)$ be a homogeneous incompressible Newtonian micropolar fluid subject to $(f, \tau)$ and the boundary effects $(f_b, t_b)$ such that

- the fluid is purely mechanical,
- the stress tensor $T$ and couple stress tensor $M$ are frame-invariant, and
- the Onsager reciprocity relations are satisfied.
Then the Eulerian velocity \( u \), pressure \( p \), Eulerian microinertia \( J \), and Eulerian angular velocity \( \omega \) satisfy

\[
\begin{cases}
\rho(\partial_t u + u \cdot \nabla u) = (\mu + \kappa/2)\Delta u + \kappa \nabla \times \omega - \nabla p + f \text{ in } \Omega(t), \\
\nabla \cdot u = 0 \text{ in } \Omega(t), \\
J(\partial_t \omega + u \cdot \nabla \omega) + \omega \times J\omega = \kappa \nabla \times u - 2\kappa \omega + (\alpha + \beta/3 - \gamma)\nabla(\nabla \cdot \omega) + (\beta + \gamma)\Delta \omega + \tau \text{ in } \Omega(t), \\
\partial_t J + u \cdot \nabla J - [\Omega, J] = 0 \text{ in } \Omega(t), \\
\mu(\nabla u)n + \kappa((1/2)\nabla \times u - \omega) \times n - \kappa n = f_b \text{ on } \partial\Omega(t), \text{ and} \\
\alpha(\nabla \cdot \omega)n + \beta(\nabla^0 \omega)n + \gamma(\nabla \times \omega) \times n = \tau_b \text{ on } \partial\Omega(t)
\end{cases}
\]

for some mass density \( \rho > 0 \), dissipation coefficients \( \mu, \kappa, \alpha, \beta, \gamma \geq 0 \), and where \( n \) denotes the outer unit normal to \( \Omega(t) \).

**Proof.** It follows from the incompressibility, Corollary 4.10, Proposition 4.17, that \( (u, \omega, J) \) satisfy

\[
\begin{cases}
\rho(\partial_t u + u \cdot \nabla u) = \nabla \cdot T + f, \nabla \cdot u = 0, \\
J(\partial_t \omega + u \cdot \nabla \omega) + \omega \times J\omega = 2 \text{ vec } T + \nabla \cdot M + \tau, \text{ and} \\
\partial_t J + \nabla \cdot (J \otimes u) - [\Omega, J].
\end{cases}
\]

In particular since \( u \) is divergence-free we see immediately that

\[
\partial_t J + \nabla \cdot (J \otimes u) - [\Omega, J] = \partial_t J + u \cdot \nabla J - [\Omega, J].
\]

Now we deduce from Theorem 5.20 that

\[
T = \mu \nabla u + \kappa \text{ ten } \left( \frac{1}{2} \nabla \times u - \omega \right) - \tau I \text{ and } M = \alpha(\nabla \cdot \omega)I + \beta \nabla^0 \omega + \gamma \text{ ten } \nabla \times \omega
\]

such that, in light of Lemma 6.16, Lemma 6.12, and the incompressibility,

\[
\nabla \cdot T = \mu \nabla \cdot \nabla u + \kappa \text{ ten } \left( \frac{1}{2} \nabla \times u - \omega \right) - \nabla p
\]

\[
= \mu \Delta u - \kappa \nabla \times \left( \frac{1}{2} \nabla \times u - \omega \right) - \nabla p
\]

\[
= \left( \mu + \frac{\kappa}{2} \right) \Delta u + \kappa \nabla \times \omega - \nabla p
\]

whilst

\[
2 \text{ vec } T = 2 \text{ vec } \left( \kappa \text{ ten } \left( \frac{1}{2} \nabla \times u - \omega \right) \right) = \kappa \nabla \times u - 2\kappa \omega,
\]

and therefore

\[
\nabla \cdot M = \alpha \nabla (\nabla \cdot \omega) + \beta \nabla \cdot (\nabla^0 \omega) + \gamma \nabla \cdot \text{ ten } \nabla \times \omega
\]

\[
= \alpha \nabla \nabla \cdot \omega + \beta \left( 1 - \frac{2}{n} \right) \nabla (\nabla \cdot \omega) + \beta \Delta \omega - \gamma \nabla \times \nabla \times \omega
\]

\[
= \left( \alpha + \frac{\beta}{3} \right) \nabla (\nabla \cdot \omega) + \beta \Delta \omega - \gamma \nabla \nabla \cdot \omega + \gamma \Delta \omega
\]

\[
= \left( \alpha + \frac{\beta}{3} - \gamma \right) \nabla (\nabla \cdot \omega) + (\beta + \gamma) \Delta \omega.
\]

To conclude we combine Proposition 4.23 and Theorem 5.20 to obtain the boundary conditions. \( \square \)

6. Appendix

In this section we record various results that are either well-known or elementary, but which are nonetheless useful elsewhere in this chapter. In Section 6.1 we record identities from calculus and linear algebra, in Section 6.2 we record results related to ten and vec, and in Section 6.3 we record some elementary results related to matrix groups and Lie groups.
6.1. Identities from linear algebra and calculus. First we record the well-known \( \epsilon - \delta \) identity.

**Lemma 6.1. (Epsilon-delta identities)**
The following identities hold: 
\[
\epsilon_{ijl} \epsilon_{kla} = \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl} \quad \text{and} \quad \epsilon_{iab} \epsilon_{jab} = 2 \delta_{ij}.
\]

We then record a vectorized version of the \( \epsilon - \delta \) identity.

**Lemma 6.2. (Vectorized epsilon-delta identities)**
Let \( a, b, c \in \mathbb{R}^3 \). Then \( a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \).

**Proof.** This is nothing more than the ‘vectorized’ version of an epsilon-delta identity of Lemma 6.1:
\[
(a \times (b \times c))_i = \epsilon_{ijk} a_j (b \times c)_k = \epsilon_{ijk} \epsilon_{lmk} a_j b_l c_m = (\delta_{im} \delta_{jk} - \delta_{ij} \delta_{km}) a_j b_l c_m = ((a \cdot c)b - (a \cdot b)c)_i.
\]

We now record a characterization of orthogonal projections in \( \mathbb{R}^3 \) using cross-products.

**Proposition 6.3.** Let \( e, v \in \mathbb{R}^3 \) with \( e \) a unit vector. We can write the orthogonal decomposition of \( v \) with respect to \( e \) conveniently as
\[
v = (v \cdot e) e - e \times (e \times v)
\]
such that
\[
|v|^2 = |v \cdot e|^2 + |e \times v|^2.
\]

More generally, for \( v, w \in \mathbb{R}^3 \),
\[
v = \left( v \cdot \frac{w}{|w|} \right) \frac{w}{|w|} - \frac{w}{|w|} \times \left( \frac{w}{|w|} \times v \right)
\]
and \( |v|^2 |w|^2 = |v \cdot w|^2 + |v \times w|^2 \).

**Proof.** It suffices to show that \( -e \times (e \times \cdot) = (I - e \otimes e) \) since the latter is precisely the orthogonal projection unto the orthogonal complement of \( e \). This identity follows from Lemma 6.1:
\[
(-e \times (e \times v))_i = -\epsilon_{ijk} \epsilon_{lmk} e_j e_i v_m = (\delta_{im} \delta_{jk} - \delta_{ij} \delta_{km}) e_j e_i v_m = -(e_j e_i v_j - e_j e_j v_i) = (v - (v \cdot e))_i
\]
i.e. indeed \( -e \times (e \times v) = (I - e \otimes e)v \).

We now record the frame-invariance of the cross product, which is not surprising since cross products are characterized in terms of lengths and angles, both of which are preserved by orientation-preserving orthogonal transformations.

**Lemma 6.4. (Frame-invariance of the cross product)**
Let \( Q \) be an orthogonal transformation of \( \mathbb{R}^3 \), i.e. \( Q \in O(3) \). Then, for all \( v, w \in \mathbb{R}^3 \),
\[
Q(v \times w) = (\det Q)(Qv) \times (Qw).
\]

**Proof.** The key observation is that for any \( u, v, w \in \mathbb{R}^3 \), \( \det(Qu \ | \ Qv \ | \ Qw) = (\det Q) \det(u \ | \ v \ | \ w) \) since \( \det Q = 1 \) if \( Q \) is orientation-preserving and \( \det Q = -1 \) if \( Q \) is orientation-reversing. So, for all \( u, v, w \in \mathbb{R}^3 \),
\[
u \cdot Q(v \times w) = (Q^T u) \cdot (v \times w) = \det(Q^T u \ | \ v \ | \ w) = (\det Q^T) \det(u \ | \ Qv \ | \ Qw) = (\det Q) u \cdot ((Qv) \times (Qw))
\]
such that indeed the identity holds.

We now record an elementary computation dealing with outer products and matrix multiplication.

**Lemma 6.5.** Let \( A \in \mathbb{R}^{r \times s} \), \( B \in \mathbb{R}^{t \times u} \), \( v \in \mathbb{R}^s \) and \( w \in \mathbb{R}^t \). Then \((Av) \otimes (wB) = A(v \otimes w) B\).

**Proof.** This is the result of an immediate computation:
\[
((Av) \otimes (wB))_{ij} = A_{ik} v_k w_l B_{lj} = A_{ik} (v \otimes w)_{kl} B_{lj} = (A(v \otimes w) B)_{ij}.
\]

We now record a characterization of the kernel of the symmetrized gradient.

**Lemma 6.6. (Characterization of the kernel of the symmetrized gradient)**
Let \( U \subseteq \mathbb{R}^n \) be a bounded open set with Lipschitz boundary. Then
\[
\ker D \cong \mathbb{R}^n \times \text{Skew}(n)
\]
i.e. for every \( v \in H^1(U) \), \( Dv = 0 \) if and only if \( v(x) = \tilde{v} + \tilde{\Omega} x \) for some \( \tilde{v} \in \mathbb{R}^n \) and some \( \tilde{\Omega} \in \text{Skew}(n) \).
Let \( v(x) = \bar{\Omega}x \) with \((v) \in \mathbb{R}^n\) and \(\bar{\Omega} \in \text{Skew}(n)\), then \(\mathbb{D}v = \text{Sym}(\nabla v) = \text{Sym}(\bar{\Omega}) = 0\).

Conversely, suppose that \(v \in \ker \mathbb{D}\), and define
\[
\tilde{v} := \int v \quad \text{and} \quad \bar{\Omega} := \int \text{Skew}(\nabla v)
\]
such that \((x \mapsto \tilde{v} + \bar{\Omega}x) = \text{proj}_{\ker \mathbb{D}} v\). Let \(w := v - (\tilde{v} + \Omega \cdot )\) and observe that \(\mathbb{D}w = \mathbb{D}v = 0\). Crucially, note that
\[
\int w = 0 \quad \text{and} \quad \int \text{Skew}(\nabla w) = 0.
\]

Using the fundamental theorem of calculus to expand \(w\) about its average we deduce that \(w = 0\). So indeed \(v(x) = \tilde{v} + \bar{\Omega}x\).\(\square\)

We prove a “Leibniz Rule” for the material derivatives introduced in Definition 2.5.

**Lemma 6.7.** (*Leibniz rule* for material derivatives)
Let \(u\) be a vector field and let \(T\) and \(S\) be differentiable tensor fields. Then \(\mathbb{D}^u_T (T \cdot S) = (\mathbb{D}^u_T T) \cdot S + T \cdot (\mathbb{D}^u_T S)\).

**Proof.** For better readability despite the number of indices involved, we will write \(I\) instead of \(i_1 \ldots i_k\) and \(J\) instead of \(j_1 \ldots j_l\). We then compute:
\[
\mathbb{D}^u_T (T \cdot S) = \partial_i (T \cdot S) + \nabla \cdot ((T \cdot S) u)
\]
where
\[
(\nabla \cdot ((T \cdot S) u))_{IJ} = \partial_a (T_{ib} S_{bJ} u_a) = \partial_a (T_{ib} u_a) S_{bJ} + T_{ib} u_a \partial_a S_{bJ} + (\nabla \cdot (T \otimes u))_{ib} S_{bJ} + T_{ib} ((u \cdot \nabla) S)_{bJ}
\]
and hence
\[
\mathbb{D}^u_T (T \cdot S) = (\partial_i T) \cdot S + T \cdot (\partial_i S) + (\nabla \cdot (T \otimes u)) \cdot S + T \cdot ((u \cdot \nabla) S) = (\mathbb{D}^u_T T) \cdot S + T \cdot (\mathbb{D}^u_T S) .
\]
\(\square\)

We record a differential identity useful when deriving the local version of the balance of energy.

**Lemma 6.8.** Let \(M\) and \(v\) be differentiable matrix and vector fields respectively, i.e. \(M : \mathbb{R}^n \to \mathbb{R}^{k \times n}\) and \(v : \mathbb{R}^n \to \mathbb{R}^n\). Then \(\nabla \cdot (v \cdot M) = (\nabla \cdot M) \cdot v + M : \nabla v\).

**Proof.** We compute: \(\nabla \cdot (v \cdot M) = \partial_i (v_j M_{ji}) = M_{ji} \partial_i v_j + (\partial_i M_{ji}) v_j = M : \nabla v + (\nabla \cdot M) \cdot v\).\(\square\)

Here we define the deviatoric part of a matrix. Due to the fact that this plays well with an orthogonal decomposition of \(\mathbb{R}^{n \times n}\), this comes in handy when establishing sign conditions on the coefficients that arise in the constitutive equations for the stress and couple stress tensors.

**Definition 6.9.** (Deviatoric part)
We define \(\text{Dev} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}\) via, for every \(M \in \mathbb{R}^{n \times n}\), \(\text{Dev} M := \text{Sym}(M) - \frac{1}{n} \text{tr} M I\), i.e. \(\text{Dev} M\) is the traceless symmetric part of \(M\).

We also define the related notion of the deviatoric gradient.

**Definition 6.10.** (Deviatoric gradient)
Let \(v\) be a differentiable vector field on \(\mathbb{R}^n\). Its **deviatoric gradient**, denoted \(\mathbb{D}^0 v\), is defined to be the deviatoric part of the symmetrized gradient of \(v\), i.e. \(\mathbb{D}^0 v := \text{Dev}(\mathbb{D} v)\).

Now we take note of how to relate the deviatoric gradient to the symmetrized gradient.

**Lemma 6.11.** (Relating the symmetrized gradient and the deviatoric gradient)
For any differentiable \(n\)-dimensional vector field \(v\), i.e. \(v \in C^1(\mathbb{R}^n, \mathbb{R}^n)\), \(\mathbb{D}^0 v = \mathbb{D} v - \frac{2}{n} (\nabla \cdot v) I\).

We conclude this section with elementary identities involving various first order differential operators.

**Lemma 6.12.** (Identities involving the curl, divergence, symmetrized gradient, and deviatoric gradient).
For any twice-differentiable \(n\)-dimensional vector field \(v\), i.e. \(v \in C^2(\mathbb{R}^n, \mathbb{R}^n)\), the following hold:
\[\begin{align*}
(1) \quad \nabla \times (\nabla \times v) &= \nabla (\nabla \cdot v) - \Delta v \quad \text{(when } n = 3) \\
(2) \quad \nabla \cdot (\mathbb{D} v) &= \nabla (\nabla \cdot v) + \Delta v
\end{align*}\]
\[
\n(3) \quad \nabla \cdot (D^0 v) = (1 - \frac{2}{n}) \nabla (\nabla \cdot v) + \Delta v
\]

\textbf{Proof.} These identities follow from direct computations. To obtain (1) we use Lemma 6.1:
\[
(\nabla \times (\nabla \times v))_i = \epsilon_{kij} \partial_j (\epsilon_{klm} \partial_m v_l) = (\delta_{lj} \delta_{jm} - \delta_{lm} \delta_{jl}) (\partial_j \partial_m v_m) = \partial_i (\partial_j v_j) - \partial_j (\partial_i v_i) = (\nabla (\nabla \cdot v) - \Delta v)_i.
\]
(2) and (3) then follow from direct computations:
\[
(2) \quad (\nabla \cdot (Dv))_i = \partial_j (\partial_i v_j + \partial_j v_i) = \partial_j (\partial_j v_i) = (\Delta v + \nabla (\nabla \cdot v))_i
\]
and
\[
(3) \quad \nabla \cdot (D^0 v) = \nabla \cdot (\text{Dev} (Dv)) = \nabla \cdot (Dv - \frac{2}{n} (\nabla \cdot v) I) = \Delta v + \left(1 - \frac{2}{n}\right) \nabla (\nabla \cdot v).
\]

\section{6.2. Skew-symmetric matrices in three dimensions.}

In this section we obtain various results that have to do with \textit{ten} and \textit{vec}. The first result is the most important one, showing that \textit{ten} and \textit{vec} are linear isomorphisms and can thus indeed be used to identify \text{Skew}(3) and \mathbb{R}^3.

\textbf{Proposition 6.13.} \textit{(Isomorphism between 3-by-3 skew-symmetric matrices and 3-dimensional vectors)}
\[
\text{vec} : \text{Skew}(3) \to \mathbb{R}^3 \text{ is an isomorphism, whose inverse is } \text{ten}, \text{ such that for every } \Omega \in \text{Skew}(3), \Omega = \text{vec}(\Omega) \times, \text{ i.e. the action of skew-symmetric matrices is equivalent to the action of a vectors via the cross-product.}
\]

\textbf{Proof.} It suffices to show that \textit{vec} and \textit{ten} from Definition 2.18 satisfy
\[
\text{vec} \circ \text{ten} = \text{id}_{\text{Skew}(3)} \text{ and } \text{ten} \circ \text{vec} = \text{id}_{\mathbb{R}^3}
\]
and that, for every \( \Omega \in \text{Skew}(3) \) and every \( v \in \mathbb{R}^3, \Omega v = \text{vec}(\Omega) \times v \). The first two identities follow from the epsilon-delta identities in Lemma 6.1. Indeed we may compute that
\[
\text{vec}(\text{ten}(\omega))_i = \frac{1}{2} \epsilon_{ab} \text{ten}(\omega)_{ab} = \frac{1}{2} \epsilon_{ab} \epsilon_{ajb} \omega_j = \frac{1}{2} (2 \delta_{ij}) \omega_j = \omega_i
\]
and
\[
\text{ten}(\text{vec}(\Omega))_{ij} = \epsilon_{iaj} \left( \frac{1}{2} \epsilon_{kai} \Omega_{kl} \right) = \frac{1}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \Omega_{kl} = \text{Skew}(\Omega)_{ij} = \Omega_{ij}
\]
Similarly, a direct computation shows that
\[
((\text{vec} \Omega) \times v)_i = \epsilon_{ijk} (\text{vec} \Omega)_j v_k = \frac{1}{2} \epsilon_{ijk} \epsilon_{ajb} \Omega_{ab} v_k = \frac{1}{2} (\delta_{ia} \delta_{kb} - \delta_{ib} \delta_{ka}) \Omega_{ab} v_k = \frac{1}{2} (\Omega_{ik} - \Omega_{ki}) v_k
\]
i.e. \( (\text{vec} \Omega) \times v = (\text{Skew} \Omega) v = \Omega v \). \hfill \Box

We now record the fact that \textit{ten} and \textit{vec} are frame-invariant.

\textbf{Lemma 6.14.} \textit{(Frame-invariance of ten and vec)}
\textit{For any } \textit{v} \textit{\in } \mathbb{R}^3, \textit{any 3-by-3 skew-symmetric matrix } \textit{A}, \textit{and any orthogonal transformation } \textit{Q},
\[
\text{ten}(Qv) = (\det Q) Q(\text{ten} v) Q^T \text{ and } \text{vec} QAQ^T = (\det Q) \text{vec} A.
\]

\textbf{Proof.} This is an immediate consequence of Lemma 6.4 since it allows us to compute that, for any \( w \in \mathbb{R}^3, \)
\[
\text{ten}(Qv)w = (Qv) \times w = (\text{det Q}) Q(\text{v} \times (Q^T w)) = (\text{det Q}) Q(\text{ten} v) Q^T w
\]
and
\[
\text{vec}(QAQ^T) \times w = QAQ^T w = Q(\vec{A} \times Q^T w) = (\text{det Q})(\text{vec} A) \times w.
\]
\hfill \Box

Here we record how \textit{ten} and \textit{vec} relate to \textit{Skew}, the linear operator which isolates the skew-symmetric part of a matrix.

\textbf{Lemma 6.15.} \textit{(Relation between ten, vec, and Skew)} The following identity holds: \textit{ten} \circ \textit{vec} = \text{Skew}.

\textbf{Proof.} This identity is nothing more than the classical epsilon-delta identity (c.f. Lemma 6.1) in coordinate-invariant-form. Indeed, let \( A \) be a real \( n \)-by-\( n \) matrix. Then using Lemma 6.1 and Proposition 6.13 we compute that
\[
\text{ten}(\text{vec} A)_{ij} = \epsilon_{iaj} \text{vec} A_{a} = \frac{1}{2} \epsilon_{iaj} \epsilon_{pqa} A_{pq} = \frac{1}{2} (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) A_{pq} = \frac{1}{2} (A_{ij} - A_{ji}) = (\text{Skew} A)_{ij}.
\]
\hfill \Box
We now derive the relationship between ten, vec, and various first order differential operators.

**Lemma 6.16. (Relation of ten, vec, Skew, and differential operators)**

For any 3-dimensional differentiable vector field \( v \), i.e. \( v : \mathbb{R}^3 \to \mathbb{R}^3 \), the following identities hold:
\[
\text{vec} \, \nabla v = \frac{1}{2} \nabla \times v, \quad \text{Skew} \, \nabla v = \frac{1}{2} \text{ten} \, \nabla \times v, \quad \text{and} \quad \nabla \cdot (\text{ten} \, v) = -\nabla \times v.
\]

**Proof.** Recall that \((\nabla \times v)_i = \epsilon_{ijk} \partial_j v_k\). We may thus compute that
\[
(\text{vec} \, \nabla v)_i = \frac{1}{2} \epsilon_{ijk} \partial_j v_k = \frac{1}{2} \nabla \times v \quad \text{and} \quad (\nabla \cdot (\text{ten} \, v))_i = \partial_j (\text{ten} \, v)_{ij} = \partial_j (\epsilon_{iaj} v_a) = -\nabla \times v
\]
such that the first and third identities hold. The second identity follows from combining the first identity and Lemma 6.15 since \text{Skew} \, \nabla v = \text{ten} \, \nabla v = \frac{1}{2} \text{ten} \, \nabla \times v. \quad \square

We conclude this section with a result relating ten, vec, and inner products.

**Lemma 6.17. (Relation between ten, vec, and inner products)**

For any \( v \in \mathbb{R}^3 \) and any \( M \in \mathbb{R}^{3 \times 3} \), \((\text{vec} \, M) \cdot v = \frac{1}{2} M : (\text{ten} \, v)\).

**Proof.** We compute:
\[
(\text{vec} \, M) \cdot v = (\text{vec} \, M)_i v_i = \frac{1}{2} \epsilon_{iab} M_{ab} v_i = \frac{1}{2} M_{ab} (\text{ten} \, v)_{ab} = \frac{1}{2} M : (\text{ten} \, v). \quad \square
\]

### 6.3. Matrix groups, and Lie groups

In this section we prove some elementary results concerning matrix groups and Lie groups. We begin by showing that the space of skew-symmetric matrices is closed under conjugacy by orthogonal matrices.

**Proposition 6.18. (Closure of skew-symmetric matrices under conjugacy by orthogonal matrices)**

\(\text{Skew} \, n\) is closed under conjugacy by \(O \, n\), i.e. for any \( A \in \text{Skew} \, n \) and \( R \in O \, n \), \( R A R^{-1} \in \text{Skew} \, n \).

**Proof.** This is immediate: \((R A R^{-1})^T = R^{-T} A^T R^T = R(-A) R^{-1} = -(R A R^{-1})\). \quad \square

We now record a result dealing with matrix groups, identifying the tangent space to the Lie group of orthogonal matrices.

**Proposition 6.19. (Tangents to matrix groups, or a glimpse into Lie algebras)**

1. \(T_I O \, n = \text{Skew} \, n\), i.e. the Lie algebra of orthogonal matrices consists precisely of the algebra of skew-symmetric matrices.
2. For any \( R \in O \, n \), \( T_R O \, n = R \text{Skew} \, n \).

**Proof.**

1. Consider \( R : \mathbb{R} \to O \, n \) such that \( R(0) = I \). Upon differentiating \( I = RR^T \) and evaluating at \( 0 \), we obtain that \( 0 = \dot{R}(0) R(0)^T + R(0) \dot{R}(0)^T = \dot{R}(0) + \dot{R}(0)^T \) i.e. indeed \( \dot{R}(0) \in \text{Skew} \, n \). Conversely, for any \( A \in \text{Skew} \, n \), define \( R(t) := e^{At} \) and observe that
\[
(e^{At})^T e^{At} = e^{AT} e^{At} = e^{(A^T+A)t} = e^0 = I \quad \text{and} \quad (e^{At})'_{t=0} = Ae^0 = A
\]
such that indeed \( \text{Skew} \, n \subseteq T_I O \, n \) and so (1) holds.

2. Fix \( R_0 \in O \, n \) and consider \( R : \mathbb{R} \to O \, n \) such that \( R(0) = R_0 \). Note that \( L_{R_0^{-1}} \circ R \) is a path along \( O \, n \) going through the identity at time zero, and hence by the above:
\[
d \left( L_{R_0^{-1}} \circ R \right)_{t=0} =: A \in \text{Skew} \, n
\]
Now, since the exterior derivative 'commutes' with composition, we have that
\[
A = d \left( L_{R_0^{-1}} \circ R \right) = dL \circ dR = R_0^{-1} \circ \dot{R}
\]
i.e. \( \dot{R} = R_0 A \in R_0 \text{Skew} \, n \).
Conversely, for any \( R_0 \in O \, n \) and \( A \in \text{Skew} \, n \), define \( R(t) := R_0 e^{At} \) such that
\[
(R_0 e^{At})^T R_0 e^{At} = e^{AT} R_0 R_0 e^{At} = e^{(A^T+A)t} = I \quad \text{and} \quad (R_0 e^{At})'_{t=0} = R_0 Ae^0 = R_0 A
\]
such that indeed \( R_0 \text{Skew} \, n \subseteq T_{R_0} O \, n \) and thus (2) holds. \quad \square
We now prove results about Lie groups and the determinant in particular. The goal of these results is to justify in a clean way the formula for the derivative of the determinant. First we prove that, in order to compute the derivative of a Lie group homomorphism (such as the determinant), it suffices to compute its derivative at the identity.

**Proposition 6.20. (Differential of a Lie group homomorphism)**
Let $G, H$ be Lie groups and let $F : G \to H$ be a Lie group homomorphism. Recall that for any $g \in G$, $L_g$ denotes left-multiplication, i.e. $L_g(h) = gh$ for any $h \in G$. Then, for any $g \in G$,

$$dF_g = dL_{F(g)} \circ dF_e \circ dL_{g^{-1}}$$

or, written in all its gory detail:

$$dF_g |_{g} = dL_{F(g)} F(g) |_{e_H} \circ dF_e |_{G} \circ dL_{g^{-1}} |_{g}$$

Moral: To compute the differential of a Lie group homomorphism it is enough to know how to compute its differential at the identity and the differentials of the left-multiplication operators.

**Proof.** Since $F$ is a group homomorphism, the following holds for any $g \in G$:

$$F = L_{F(g)} F \circ L_g^{-1}$$

Therefore, upon applying the exterior derivative and noting that it ‘commutes’ with composition, we obtain the desired result. □

Following Proposition 6.20, since we are seeking a formula for the derivative of the determinant we compute its derivative at the identity.

**Lemma 6.21. (Derivative of the determinant at the identity)**

$$\det' |_I = \text{tr}$$

**Proof.** Let $H$ be an arbitrary $n$-by-$n$ matrix which we write as $H = (h_1 h_2 \ldots | h_n)$. Then

$$\det (I + \epsilon H) = \det (e_1 + \epsilon h_1 | \ldots | e_n + \epsilon h_n)$$

$$= \det (e_1 | \ldots | e_n) + \epsilon (\det (h_1 | \ldots | e_n) + \cdots + \det (e_1 | \ldots | h_n)) + O (\epsilon^2)$$

$$= 1 + \epsilon (h_{11} + \cdots + h_{nn}) + O (\epsilon^2)$$

and hence

$$\det' |_I (H) = \lim_{\epsilon \to 0} \frac{\det (I + \epsilon H) - \det I}{\epsilon} = \text{tr} H$$

□

Finally we conclude this section by using Proposition 6.20 and Lemma 6.21 to establish a formula for the derivative of the determinant.

**Corollary 6.22. (Derivative of the determinant)**

Let $A$ and $M_0$ be $n$-by-$n$ matrices, with $M_0$ invertible. Then

$$\det' |_{M_0} (A) = \det (M_0) \text{tr} (M_0^{-1} A)$$

**Proof.** Since the determinant is a Lie group homomorphism from $GL(n)$ to $\mathbb{R}$ (as a multiplicative group), we know by Proposition 6.20 that it is enough to compute the derivative of the determinant at the identity as well as the derivative of the left-multiplications. By Lemma 6.21, $\det' |_I = \text{tr}$, and since here both left-multiplications are linear (and hence equal to their derivatives), we obtain that:

$$\det' |_{M_0} (A) = \left( L_{\det(M_0)} \circ \text{tr} \circ L_{M_0^{-1}} \right) (A) = \det (M_0) \det (M_0^{-1} A)$$

□
Bibliography


