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STRICT SIGN-CENTRAL MATRICES*

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Abstract. The notions of *central* and *sign-central* matrices have been studied for some time in the literature. In this paper, we introduce and study some similar, but stronger, notions: *strict central* and *strict sign-central* matrices. The investigation is motivated by some basic questions in mathematical finance, and this connection is discussed.

Key words. qualitative matrix theory, sign-central matrices, mathematical finance

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1. Introduction.

A vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is *nonnegative* (resp., *nonpositive*, *positive*) if each component x_i is nonnegative (resp., nonpositive, positive). The *sign matrix* of a matrix A, denoted by sign A, is the matrix obtained by replacing each entry in A by its sign: 0, 1, or -1. The qualitative class of A consists of all matrices with the same sign matrix as A. A matrix A is *central* if there is a nonzero nonnegative vector in its null space. If each matrix in the qualitative class of A is central, then A is called *sign-central*. Both of these notions were investigated in [2], [6], and an extension in [7].

In this paper, we study stronger requirements than being central or sign-central. Let A be a (real) matrix. We say that A is a *strict central* matrix if A has a positive vector in its null space. Moreover, A is a *strict sign-central* matrix, or an *SSC matrix*, if each matrix in the qualitative class of A is strict central. Clearly, if a matrix is SSC, then it is also SC. A referee kindly pointed out the paper [9], where an essentially equivalent concept was studied in connection with sign-solvability of linear system of equations; see later discussions.

Qualitative matrix problems have been studied for some time in linear algebra. In particular, the book [6] investigates in detail properties of sign-solvable linear systems and related issues. The notion of sign-solvability has its origins in economics, where one wants to answer qualitative stability properties of the equilibrium of an economic system. A motivation for the present study of SSC matrices lies in the area of mathematical finance.

The paper is organized as follows. In section 2, a characterization of strict central matrices is given, and the motivation in mathematical finance is presented. Some general properties of strict sign-central matrices are treated in section 3, and a certain subclass is introduced. Section 4 contains a characterization of SSC matrices, along with some consequences, e.g., in mathematical finance. In section 5, we prove an upper bound on the number of columns of minimal SSC matrices, and in the final section we study classes of minimal SC and SSC matrices.

Notation. Vectors are column vectors, and they are identified with corresponding n-tuples. $M_{m,n}$ is the set of all real $m \times n$ matrices. I_n is the identity matrix of order

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n. For a matrix A, $\mathrm{Nul}\,(A)$, $\mathrm{Row}\,A$, and $\mathrm{Col}\,A$ denote its null space, row space, and column space, respectively. The jth column of A is denoted by $A^{(j)}$. e denotes the all ones vector (of suitable dimension). $\mathbb{R}^n_+$ is the set of nonnegative vectors in $\mathbb{R}^n$. $L^\perp$ denotes the orthogonal complement of a subspace L in $\mathbb{R}^n$. Two matrices $A, B \in M_{m,n}$ are called *qualitatively equivalent* whenever $\mathrm{sign}\,A = \mathrm{sign}\,B$. The *qualitative class* of A is defined by $\mathcal{Q}(A) = \{B \in M_{m,n} : \mathrm{sign}\,A = \mathrm{sign}\,B\}$. The convex hull of a set $S \subseteq \mathbb{R}^n$ is denoted by $\mathrm{conv}\,S$ (see [14]). The *support* of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is defined as $\mathrm{supp}(x) = \{i \le n : x_i \neq 0\}$.

2. Strict central matrices and an application. Let $A \in M_{m,n}$ and denote its jth column by $A^{(j)}$ $(j \le n)$. Define the associated polytope

$$\mathcal{C}(A) = \mathrm{conv}\,\{A^{(1)}, A^{(2)}, \ldots, A^{(n)}\} = \left\{ \sum_{j=1}^{n} \lambda_j A^{(j)} : \lambda_j \ge 0 \;\; (j \le n), \;\; \sum_j \lambda_j = 1 \right\}.$$

By considering a vector in the null space of A and using suitable scaling, it follows that A is central if and only if the origin O lies in $\mathcal{C}(A)$. Moreover, A is strict central if and only if O is a relative interior point of $\mathcal{C}(A)$, i.e., it is an interior point with respect to the relative topology induced by the affine hull of $\mathcal{C}(A)$.

The property of being sign-central is preserved under permutation of rows or columns.

For given $A \in M_{m,n}$, define

$$\gamma(A) = \sup\{e^T y : y = A^T x,\, y \ge O,\, x \in \mathbb{R}^m\}. \tag{2.1}$$

In this supremum, the vector $y = A^T x$ runs through the row space of A (where we let this consist of column vectors), and $\gamma(A)$ is either 0 or $+\infty$. Computationally, $\gamma(A)$ may be found by linear programming (LP), where $\gamma(A) = \infty$ means that the linear optimization problem in (2.1) is unbounded. The class of strict central matrices may be characterized as follows.

THEOREM 2.1. *Let $A \in M_{m,n}$. Then the following statements are equivalent:*
(i) *A is a strict central matrix.*
(ii) *The only nonnegative vector in* $\mathrm{Row}\,A$ *is the zero vector.*
(iii) $\gamma(A) = 0$.

Proof. (i) $\Rightarrow$ (ii). Assume that A is strict central, and let z be a positive vector in the null space of A. So $Az = O$, and therefore, for any $w \in \mathbb{R}^m$, $0 = w^T(Az) = (w^T A)z$. But, as z is positive, the vector $A^T w$ cannot be both nonnegative and nonzero. Therefore, (ii) holds.

(ii) $\Rightarrow$ (iii). Assume that (ii) holds. Then, clearly, $\gamma(A) = 0$, so (iii) holds.

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Since (2.1) may be viewed as a linear optimization problem (which has a feasible solution, i.e., a point satisfying the constraints), we may apply the linear programming duality theorem [12]: the dual problem is

$$\min\{0 : Az = O,\, z = w + e,\, w \ge O,\;\, z, w \in \mathbb{R}^n\},$$

and this minimum is also 0, i.e., the problem has a feasible solution. In particular, $z = w + e \ge e$. But $z \in \mathrm{Nul}\,(A)$ and z is a positive vector, so A is strict central and (i) holds. $\square$

This theorem is essentially a consequence of *Stiemke's theorem* (see [12]), which says the following: there exists a positive vector x with $Ax = O$ if and only if for all vectors y, $y^T A \le O$ implies that $y^T A = O$. The last property is clearly equivalent to

(ii) of our theorem. These results are closely connected to Farkas's lemma and linear programming duality; see [12].

We now discuss a connection to mathematical finance which actually motivated the present study. One may consult, e.g., [8] for an introduction to mathematical finance, and [15] for mathematical finance and stochastic differential equations.

In a simple, but basic, model of a discrete *financial market*, one has n assets and m scenarios, each representing a possible development of asset values from the present time t_0 to time t_1, one time step ahead. The financial market is represented by a matrix $P = [p_{ij}] \in M_{m,n}$, where p_{ij} is the relative change in the value of the jth asset j under the ith scenario. Here, p_{ij} may be positive, negative, or zero. A *portfolio* is a vector $x \in \mathbb{R}^n$ whose jth component x_j is the quantity of asset j an investor holds from time t_0 to t_1. A negative component of x corresponds to a short position. For a portfolio x, the vector Px represents the payoff of the portfolio under each of the scenarios. If Px is positive, i.e., $(Px)_i > 0$ for each i, it means that one makes money in every scenario, a very attractive situation. A weaker property is that Px is nonnegative, but nonzero, which means that one is guaranteed against any loss, and under at least one scenario one makes a profit. This last property is called an *arbitrage*. Basic economic principles say that if a financial market contains an arbitrage, trading would be initiated and prices would soon adjust so that an arbitrage opportunity disappears. A version of the fundamental theorem of asset pricing/mathematical finance [8] says that a financial market has no arbitrage if and only if there is a probability measure on the set of scenarios which makes each asset price process a martingale. The latter statement means, in this one-step discrete model, that there is a positive vector in the null space of P^T. Thus, the fundamental theorem of asset pricing/mathematical finance, in this discrete model, follows from Theorem 2.1, the equivalence of (i) and (ii) for the matrix $A := P^T$. So the market is arbitrage-free precisely when P^T is a strict central matrix.

In practice, specifying the numbers p_{ij} may be hard and based on beliefs, other models, etc. A less ambitious task is only to specify, for each scenario and asset, whether the payoff is positive, negative, or zero. This corresponds to specifying a $(0, \pm 1)$-matrix P. For instance, historical data and correlations could serve as a basis for such a qualitative specification of payoffs. To our knowledge, similar qualitative questions have not been studied before in mathematical finance. In economics, however, Paul Samuelson [11] initialized qualitative mathematical questions for economic models; see a discussion in [6].

This qualitative idea in mathematical finance may lead to several interesting questions, and a basic question is the following:

(2.2) Given a $(0, \pm 1)$-matrix P, when is the corresponding financial market arbitrage-free for all markets $\tilde{P}$ in the same qualitative class as P?

This is a qualitative robustness question concerning the financial market. Note that this concept is different from the usual principle of robustness where "small" perturbations of data results in "small" perturbations of "output." Based on the discussion above, the answer to the question in (2.2) is yes precisely when the matrix P^T is strict sign-central. Later, we characterize this property and interpret the characterization for the financial market model. Moreover, several other properties of SSC matrices are investigated. We should stress that the model of a financial market just discussed is perhaps the most basic one. When trading of assets can be made at any time within a given time interval, and the scenario space is infinite, more realistic problems can be discussed and analyzed using stochastic analysis; see, e.g., [15].

3. Strict sign-central matrices. In order to study the SSC-property, it suffices to consider $(0, \pm 1)$-matrices (i.e., each entry is 0, 1, or -1).

In [9], the following notion was introduced. A $(0, \pm 1)$-matrix A is called an L^+-*matrix* provided that each $B \in \mathcal{Q}(A)$ satisfies $\{y : y^T B \geq O\} = \{O\}$. Theorem 2.4 in [9] gives characterizations of the L^+-property, in particular that A is an L^+-matrix if and only if A has no zero rows and has a positive vector in its null space. This equivalence is nontrivial; one direction is easy to prove, but the other requires a separation theorem from convex analysis (see later). Therefore, the concept of an L^+-matrix, due to Lee and Shader, and our SSC-concept are essentially the same; the only difference is that SSC matrices may have zero rows.

A vector is *balanced* [6] if it is either the zero vector or it contains both a positive and a negative entry. A vector which is not balanced is called *unisigned*. If $A \in M_{m,n}$ is an SSC matrix, then clearly each row of A is balanced (as the null space has a positive vector).

Example 3.1. Consider the two matrices

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Both matrices have only balanced rows. A is strict central, as $Ae = O$, while B is not strict central as B is invertible (so O is the only vector in its null space). So strict centrality may vary within the same sign class of matrices. □

If A is an SSC matrix (resp., SC matrix), then each submatrix A' consisting of a subset of the rows in A is also an SSC matrix (resp., SC matrix). (This follows from the fact that, if B (resp., B') is in the sign class of A (resp., A'), then $\mathrm{Nul}\, B \subseteq \mathrm{Nul}\, B'$.)

If A is an SSC matrix of size $m \times n$, then every $B \in \mathcal{Q}(A)$ satisfies $\mathrm{rank}\, B < n$ (otherwise, $\mathrm{Nul}\, B = \{O\}$). Let $\rho(A)$ denote the *covering number* of a matrix A, i.e., the minimum number of lines (rows and/or columns) needed to cover all nonzeros of A; see [5].

THEOREM 3.2. *Let $A \in M_{m,n}$. If A is an SC matrix, then $\rho(A) < n$.*

Proof. We may assume that the SC matrix $A = [a_{ij}]$ is a $(0, \pm 1)$-matrix. By König's minmax theorem [5], $\rho(A)$ is equal to the maximum number of nonzero entries in A, no two of which are in the same line.

Suppose $\rho(A) = n$. Without loss of generality (after suitable line permutations), we may assume the main diagonal of A (n entries) consists of ± 1 (i.e., no zeros). Let $B = [b_{ij}]$ be a matrix with the same sign pattern as A such that $|b_{ii}| = K$ $(i \leq n)$ where K is a "very large" number, and $|b_{ij}|$ is a "very small" number whenever $i \neq j$ and $a_{ij} \neq 0$. Then the determinant of the leading $n \times n$ submatrix A_0 of A is not zero:

$$|\det A| \approx |K^n \pm n! K^{n-2}| > 0$$

provided that K is large enough. But then A_0 is invertible and the rank of A is n, which contradicts that A is SC. So, by contradiction, it follows that $\rho(A) < n$. □

In connection with this theorem, we remark that $\rho(A) < n$ is stronger than $\mathrm{rank}\, A < n$.

In the remaining part of this section, we discuss two special cases in which a complete, and rather simple, characterization of the SSC property may be found. An SSC matrix A is a *minimal SSC* matrix if the deletion of a column always gives a non-SSC matrix. Similarly, an SC matrix A is a *minimal SC* matrix if the deletion of a column always gives a non-SC matrix.

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

FIG. 1. *Minimal SSC matrices, $m = 2$.*

This concept of minimality is also of interest in our mathematical finance application (section 2). Assume that a matrix P represents a financial market as discussed before, and assume that P^T is SSC, so no arbitrage exists for any market represented by a matrix in the qualitative class of P. Then P^T being minimal SSC means that the deletion of any scenario (i.e., a row in P) results in a market with an arbitrage opportunity for some matrix in $\mathcal{Q}(P)$.

A matrix with a single row is SSC if and only if its row is balanced. Consider the case $m = 2$, where we may describe all SSC matrices as follows. One possibility is that one row is zero and the other is balanced. When both rows are nonzero, there are, up to symmetry, only five minimal SSC matrices, and they are shown in Figure 1. The symmetries are (i) permutation of coordinates and (ii) multiplying the matrix by a strict signing.

The second special case involves incidence matrices. In a $(0, \pm 1)$-matrix which is SSC, each nonzero row contains a $+1$ and a -1. It is therefore natural to consider the class of matrices where each row contains precisely two nonzeros, a $+1$ and a -1. This is the class of incidence matrices of directed graphs [5] where we associate rows with edges.

Let $G = (V, E)$ be a directed graph with n vertices and m (directed) edges. For an edge $e = (u, v) \in E$, we let $u = t(e)$ and $v = h(e)$ denote the tail and head of e, respectively. Let $A_G = [a_{ij}]$ denote an (oriented) incidence matrix of G. Thus, we order vertices and edges, say, the vertices are $v_1, v_2, \ldots, v_n$ and the edges $e_1, e_2, \ldots, e_m$, and then $a_{ki} = 1$ if $v_i = t(e_k)$, $a_{kj} = -1$ if $v_j = h(e_k)$, and $a_{kl} = 0$ otherwise ($k \le m$). We allow parallel edges in the digraph. Let G^* denote the undirected graph corresponding to G; this is the graph obtained from G by ignoring directions on the edges. Recall that a tree is a connected and acyclic graph. Note that if G has parallel edges, so will G^*, and two parallel edges represent a cycle (of length 2).

THEOREM 3.3. *Let $G = (V, E)$ be a directed graph. Then A_G is an SSC matrix if and only if G^* is acyclic, i.e., it is a forest.*

Proof. Let $A = A_G$ and $B \in \mathcal{Q}(A)$. By definition, B is a strict central matrix if and only if its null space contains a positive vector. For any matrix $B = [b_{ik}] \in \mathcal{Q}(A)$, there are positive numbers α_k, β_k ($k \le n$) such that the kth row of B is given by $b_{ki} = \alpha_k$, $b_{kj} = -\beta_k$, where $v_i = t(e_k)$, $v_j = h(e_k)$, and $b_{kl} = 0$ otherwise. Then the linear system $Bz = O$ becomes

$$\alpha_k z_i - \beta_k z_j = 0 \quad (v_i = t(e_k), v_j = h(e_k),\ k \le m). \tag{3.1}$$

Assume that G^* has a cycle C, say, with edges $e_{k_1}, e_{k_2}, \ldots, e_{k_t}$ (possibly, the cycle consists of two parallel edges). Consider now the specific matrix B where $\alpha_{k_1} = 2$, $\alpha_{k_2} = \alpha_{k_3} = \cdots = \alpha_{k_t} = 1$ and $\beta_{k_1} = \beta_{k_2} = \cdots = \beta_{k_t} = 1$. Then, it is easy to see that each solution z of (3.1) satisfies $z_i = 0$ for each vertex v_i in the cycle C, and,

FIG. 2. G_1^* *and* G_2^*.

furthermore, this implies that $z_i = 0$ for each vertex v_i in the connected component that contains C. This proves that if a component of G^* contains a cycle, then, for suitable $B \in \mathcal{Q}(A)$, each vector in $\mathrm{Nul}\,(B)$ has some components that are zero. This implies that B is not strict central, and therefore A is not an SSC matrix.

On the other hand, assume that G^* is acyclic, i.e., it is a forest. Let $B = [b_{ki}] \in \mathcal{Q}(A)$ be as described in the first paragraph of the proof. Consider a component of G^*. Then a small calculation shows that (3.1) has a positive solution, i.e., $z_i > 0$ for each vertex v_i in that component. This holds for every component, so the resulting vector z is positive and lies in the nullspace of B. It follows that B is strict central, and since B was arbitrary in $\mathcal{Q}(A)$, A is SSC. □

Example 3.4. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

These are the incidence matrices of directed graphs G_1 and G_2; the corresponding graphs G_1^* and G_2^* are shown in Figure 2. G_1^* is connected and contains a cycle, so A_1 is not SSC. G_2^*, however, is a tree, and therefore A_2 is SSC. Returning to the application in mathematical finance (see the introduction), there is no arbitrage in any financial market given by a matrix A whose sign pattern equals that of the matrix A_2^T above. □

Another example is the $m \times (m+1)$-matrix F_m with a 1 and a -1 in each row, shifted by one position to the right for each row

$$F_m = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}. \tag{3.2}$$

As remarked in [2], this matrix is SC. It follows from Theorem 3.3 that F_m is actually SSC as the corresponding graph G^* is a path with $m+1$ vertices.

4. Characterizations.

Observe that a matrix is SSC if and only if the matrix obtained by deleting zero columns is SSC. A zero matrix is clearly SSC. Thus, it suffices to investigate the SSC property for matrices without zero columns.

A diagonal matrix D is called a *strict signing* if its diagonal entries are ± 1. The following characterization of sign-centrality (not strict sign-centrality) was shown in [2, Theorem 2.1].

THEOREM 4.1 (see [2]). *For every $m \times n$ $(0, \pm 1)$-matrix A, the following are equivalent:*

(i) *A is a sign-central matrix.*

(ii) *For every strict signing D of order m, the matrix DA contains a nonnegative column.*

Since every SSC matrix is SC, condition (ii) in Theorem 4.1 is a necessary condition for being SSC.

Example 4.2. The matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is SC (see property (ii) in Theorem 4.1), but it is not SSC (the second row is not balanced). In other words, here the origin lies in the polytope $\mathcal{C}(B)$ for every $B \in \mathcal{Q}(A)$, but not in its (relative) interior. If, however, we delete the final column, the resulting 2×2 matrix is SSC.

Let E_m be the $m \times 2^m$ (± 1) matrix such that each m-tuple of ± 1s is a column of E_m. Then E_m is a SC matrix; in fact, E_m is a minimal SC matrix. Moreover, E_m is SSC (as we shall see below); actually, it is minimal SSC.

Note that zero rows have no affect on SC and SSC, since they do not affect the null space. A zero column in A implies that A is SC. As mentioned, a zero column in A has no effect on SSC. (The corresponding coordinate in a vector in the null space can always be taken to be positive.) So one can assume that A does not have any zero rows or zero columns.

A diagonal matrix D is called a *signing* if its diagonal entries are 0 or ± 1. So a strict signing is a special signing, but in general a signing may contain zeros on the diagonal. Clearly, if a matrix A is SSC, then DA is also SSC for every signing D; this follows easily from the definition of the SSC property.

The next theorem contains a characterization of the SSC property. Note the similarity to the characterization of the SC property in Theorem 4.1. This characterization is contained in Theorem 2.4 of [9] (see remarks before). Our proof is different, but it also uses results from convex analysis.

THEOREM 4.3. *Let A be an $m \times n$ $(0, \pm 1)$ matrix with no zero rows or columns. Then the following are equivalent:*

(i) *A is an SSC matrix.*

(ii) *For every signing $D \neq O$, the matrix DA contains a nonzero nonnegative column.*

Proof. We first show that (i) implies (ii). The proof is by induction on m. Clearly, the theorem holds for $m = 1$, so assume $m > 1$.

Assume that A is SSC and has no zero rows or columns. Let $D \neq O$ be a signing. A is also SC, so by Theorem 4.1, if D is a strict signing, then DA has a nonnegative column and since A has no zero columns, DA has a nonzero nonnegative column. Now assume that D has at least one zero. Without loss of generality, we assume that the first $1 \leq k < m$ diagonal entries of D are nonzero and the last $m - k$ equal zero. Thus,

$$DA = \left[\begin{array}{c} D_1 A_1 \\ \hline O_{m-k,n} \end{array} \right],$$

where A_1 is the leading $k \times n$ submatrix of A and D_1 is a strict signing. The matrix A_1 is an SSC matrix with no zero rows (since A has no zero rows) and with fewer rows than A. D_1A_1 may have some zero columns that we may ignore (see, the initial remark), so, by induction, D_1A_1 has a nonzero nonnegative column, and therefore DA has such a column too. This proves that (i) implies (ii).

Conversely, assume that A satisfies property (ii). Let $B = [b_{ij}] \in \mathcal{Q}(A)$. Note that B also satisfies (ii), since each column of A and the corresponding column of B lie in precisely that same orthants (as there is no sign change of any entry). For each $c \in \mathbb{R}^m$, define

$$\sigma(c) = \sup\{c^T y : y \in \mathcal{C}(B)\}.$$

Thus, σ is the support function of the polytope $\mathcal{C}(B)$ and, since $\mathcal{C}(B)$ is compact, the supremum is attained, so we could write "max" in the definition above. We now prove that the support function is positive:

$$\sigma(c) > 0 \quad \text{for each nonzero vector in } \mathbb{R}^m. \tag{4.1}$$

To prove this, let c be a nonzero vector in $\mathbb{R}^m$. Consider the signing D, given by its diagonal elements s_i, where $s_i = \text{sign}\,(c_i)$ $(i \le m)$. Then, from property (ii) applied to D and B, we conclude that DB contains a nonzero nonnegative column, say, the jth column. Thus, $s_ib_{ij} \ge 0$ for each $i \le m$ and $s_kb_{kj} > 0$ for some $k \le m$. Therefore,

$$c^T B^{(j)} = \sum_i c_ib_{ij} = \sum_{i:c_i \ne 0} c_ib_{ij} = \sum_{i:c_i\ne 0} |c_i|s_ib_{ij} > 0.$$

But then $\sigma(c) = \sup\{c^T y : y \in \mathcal{C}(B)\} \ge c^T B^{(j)} > 0$, which proves (4.1).

Next, we have the general fact from convex analysis (see [10], [14])

$$\mathcal{C}(B) = \{y \in \mathbb{R}^m : c^T y \le \sigma(c) \ \text{ for all } c \in \mathbb{R}^m\}.$$

Since $\sigma(c) > 0$ for each nonzero c, and $\mathcal{C}(B)$ is a polytope, this implies that $\mathcal{C}(B)$ is full-dimensional and O is an interior point of $\mathcal{C}(B)$. Therefore, B is SC, and A is SSC. ▯

Remarks.

1. Recall that a necessary condition for a matrix A to be SSC is that each row is balanced. This is a special case of the sign condition (ii) of Theorem 4.3 obtained from the signing D given by $s_k = \pm 1$ for some $k \le m$, and $s_i = 0$ otherwise.
2. The proof above shows that $\mathcal{C}(B)$ is full-dimensional for every $B \in \mathcal{Q}(A)$ when A is SSC and has no zero rows or columns. Of course, if one appends rows of zeros to A, the new matrix is still SSC, but then these polytopes $\mathcal{C}(B)$ are no longer full-dimensional.
3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

Every row of A is balanced, and the SC condition (ii) of Theorem 4.1 holds; this is assured by the last two columns. Actually, O is a convex combination of the last two columns of B for any $B \in \mathcal{Q}(A)$, so A is SC. However, A is not SSC, as every vector x in the null space of A satisfies $x_2 = 0$. The sign condition (ii) of Theorem 4.3 is violated by the signing D given by $s_1 = 1$,

$s_2 = 1$, $s_3 = 0$: the submatrix consisting of the first two rows of A is nonzero and has no nonnegative nonzero column.

We return to the model of financial markets discussed in section 2. Assume that the financial market is given by a $(0, \pm 1)$-matrix P of size $m \times n$, as in (2.2). For a portfolio (an investment vector) $x \in \mathbb{R}^n$, define its *payoff vector* under scenario $i \le m$ as the Hadamard product of the ith row of P and x^T, i.e.,

$$\text{Po}_i(x) = (p_{i1}x_1, p_{i2}x_2, \dots, p_{in}x_n).$$

The sign of the jth component $p_{ij}x_j$ tells whether there is positive (resp., negative, or zero) payoff from asset j under the ith scenario. If the portfolio x is a $(0, \pm 1)$-vector, we call x a *simple* portfolio, and then only unit investments are made for a selected subset of assets (possibly short positions, i.e., $x_j = -1$). From Theorem 4.3, we now obtain the following result.

THEOREM 4.4. *Consider a financial market given by a* $(0, \pm 1)$*-matrix* P *of size* $m \times n$*. Then the following are equivalent:*

(i) *Each financial market* $\tilde{P} \in \mathcal{Q}(P)$ *is arbitrage-free.*

(ii) *For each simple nonzero portfolio* x*, there is a scenario* $i \le m$ *such that its payoff vector* $\text{Po}_i(x)$ *is nonpositive and nonzero.*

Proof. As explained in section 2, property (i) holds precisely when P^T is a strict central matrix. Therefore, the present theorem is a restatement of Theorem 4.3, using the new terminology and the observation that in statement (ii) of Theorem 4.3, one may replace "nonnegative" by "nonpositive." (This corresponds to multiplying the signing by -1.) ∎

Note that statement (ii) in Theorem 4.4 is equivalent to the same statement, but the word "simple" is deleted (since this does not affect the signs in the payoff vector). We may interpret condition (ii) as follows: every (simple) portfolio has an obstruction, a scenario under which none of the assets give positive payoff, and for one of them there is loss. This explains why there is no arbitrage.

The characterizations above of the SC- and the SSC-properties may be interpreted geometrically. Each (± 1)-vector $s = (s_1, s_2, \dots, s_m)$ defines a (closed) orthant consisting of all vectors $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ with $s_i x_i \ge 0$ $(i \le m)$. Then Theorem 4.1 says that A is SC if and only if each orthant contains at least one column from A. Also, for a nonzero $(0, \pm 1)$-vector $s = (s_1, s_2, \dots, s_m)$, we define the associated *suborthant* consisting of all vectors $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ with $x_i = 0$ if $s_i = 0$, and otherwise $s_i x_i \ge 0$. Then Theorem 4.3 says that A is SSC if and only if each suborthant contains a nonzero vector which is the projection of some column of A onto that suborthant.

We now show that a consequence of Theorem 4.3 is that a nontrivial SSC matrix has more columns than rows.

COROLLARY 4.5. *Let* A *be an* $m \times n$ $(0, \pm 1)$ *matrix with no zero rows or columns. If* A *is SSC, then* $n \ge m + 1$.

Proof. As just remarked, with these assumptions, the polytope $\mathcal{C}(B)$ is full-dimensional for every $B \in \mathcal{Q}(A)$. Since the dimension of $\mathcal{C}(B)$ is at most $n - 1$, it follows that $m \le n - 1$. ∎

The matrix F_m in (3.2) is an example of an SSC matrix of size $m \times (m + 1)$. A large class of $(0, \pm 1)$ $m \times (m+1)$ SSC matrices, which contains F_m, is described next. Let $A = [a_{ij}]$ be an $m \times (m + 1)$-matrix whose ith row (for $i \le m$) consists of $i - 1$ leading zeros and has the form

$$(0, 0, \dots, 0, 1, a_{i,i+1}, a_{i,i+2}, \dots, a_{i,m+1})$$

and where $a_{ij} \le 0$ $(j > i)$ and $a_{ij} = -1$ for at least one $j > i$. Let $\mathcal{A}_m$ be the class of all matrices A constructed in this way. An example is the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

COROLLARY 4.6. *Each matrix $A \in \mathcal{A}_m$ is minimal SSC.*

Proof. This can be shown directly by a study of the nullspace of A, but it also follows from Theorem 4.3 as follows. Let D be a nonzero signing of order m, given by its diagonal elements $s_1, s_2, \ldots, s_m$. There are two cases to discuss.

Case 1. $s_i \le 0$ $(i \le m)$. Choose i as large as possible such that $s_i = -1$. Since $A \in \mathcal{A}_m$, there is a $j > i$ such that $a_{ij} = -1$. Then $DA^{(j)}$ is nonzero and nonnegative because (a) for $k < i$, $s_k a_{kj} \ge 0$ as $s_k \le 0$ and $a_{kj} \le 0$, (b) $s_i a_{ij} = (-1) \cdot (-1) = 1$, and (c) for $k > i$, $s_k a_{kj} = 0$ as $s_k = 0$.

Case 2. *There exists an i such that $s_i = 1$.* Choose i smallest possible such that $s_i = 1$. Then $DA^{(i)}$ is nonzero and nonnegative because (a) for $k < i$, $s_k a_{ki} \ge 0$ as $s_k \le 0$ and $a_{ki} \le 0$, (b) $s_i a_{ii} = 1 \cdot 1 = 1$, and (c) for $k > i$, $s_k a_{ki} = 0$ as $a_{ki} = 0$.

So it follows from Theorem 4.3 that A is SSC. Moreover, Corollary 4.5 implies that A is minimal SSC. ▯

One may therefore construct large classes of SSC matrices by choosing a matrix in $\mathcal{A}_m$ and multiplying from the left by a signing D (which, as remarked, preserves the SSC property) and finally performing row and column permutations. For instance, all SSC matrices discussed in Theorem 3.3 may be constructed in this way.

For an $m \times n$ matrix A, we define the *cone* of A to be the convex cone generated by the columns of A, i.e, $\text{cone}(A) = \{Az : z \ge O\}$. Then, if there are no zero rows in A, one can show that A is SSC if and only if $\text{cone}\,(B) = \mathbb{R}^m$ for each $B \in \mathcal{Q}(A)$; see Theorem 2.4 in [9]. Moreover, each SSC matrix with no zero rows (i.e, an L^+-matrix) is also an L-matrix, i.e., each $B \in \mathcal{Q}(A)$ has linearly independent rows. This was shown in [9], and it also follows from Theorem 4.3 combined with a result in [2] saying that a matrix with no zero rows is an L-matrix if and only if for each signing D, the matrix DA has a nonzero column which is either nonnegative or nonpositive. We remark that it is not true that an SC matrix with no zero rows or columns is an L-matrix; see [2] for a counterexample. However, *minimal* SC matrices with no zero rows are L-matrices, as shown in [2]. Clearly, there are L-matrices that are not SSC, e.g., the identity matrix.

We now characterize SSC matrices of size $m \times (m + 1)$ in terms of the reduced row echelon form.

COROLLARY 4.7. *Let A be an $m \times (m+1)$ matrix with no zero rows or columns. Then A is SSC if and only if the* (*unique*) *reduced row echelon form R of every matrix $B \in \mathcal{Q}(A)$ has the form*

$$R = \begin{bmatrix} I_m & v \end{bmatrix}, \tag{4.2}$$

where I_m is the identity matrix of order m and $v \in \mathbb{R}^m$ is a vector with only negative entries.

Proof. Assume first that A is SSC. Let $B \in \mathcal{Q}(A)$ and let $R \in M_{m,m+1}$ denote the reduced (row) echelon form of B. Since A is an L-matrix (see above), B has linearly independent rows. Thus, R contains I_m as a submatrix, and we observe that this submatrix consists of the first m columns of B. In fact, since B and R have the same null space, if the final column of R is the mth unit vector e_m, each $x = (x_1, x_2, \ldots, x_{m+1}) \in \operatorname{Nul} B$ would satisfy $x_{m+1} = 0$, contradicting that B is SSC. Thus, R has the form (4.2) for some $v \in \mathbb{R}^m$. Then $x \in \operatorname{Nul} B$ if and only if

$$x_i = -v_i\, x_{m+1} \;\; (i \le m), \text{ and } x_{m+1} \text{ is free.}$$

Since $\operatorname{Nul} B$ contains a positive vector x, this implies that $v_i < 0$ $(i \le m)$, as desired.

Conversely, assume that the reduced echelon form R of every matrix $B \in \mathcal{Q}(A)$ has the form (4.2) with each entry in v being negative. Then, using the equations just given, we see that $\operatorname{Nul} R$ contains a positive vector, e.g., $x_i = -v_i$ $(i \le m)$ and $x_{m+1} = 1$, and therefore $\operatorname{Nul} B$ contains a positive vector. This holds for every $B \in \mathcal{Q}(A)$, so A is SSC. ☐

We note that the matrices studied in Corollary 4.7 are precisely the S-matrices studied in qualitative matrix theory; see [6]. The next proposition establishes some properties of minimal SSC matrices. For vectors $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$, the *Hadamard product* $w = u \circ v$ is the vector $w = (w_1, w_2, \ldots, w_m)$, where $w_i = u_i v_i$ $(i \le m)$. The jth column of a matrix A is denoted by $A^{(j)}$. A $(0, \pm 1)$-matrix B of size $m \times k$ is called *row unisigned* provided each of its rows is unisigned. Thus, no row of a row unisigned matrix contains both a $+1$ and a -1. We then define $\epsilon(B) \in \mathbb{R}^m$ as the vector whose ith component ϵ_i is 0 if the ith row of B is zero, and otherwise ϵ_i equals the unique nonzero occurring in the ith row. (It may occur several times.)

PROPOSITION 4.8. *Let $A = [a_{ij}]$ be a $(0, \pm 1)$-matrix which is SSC. Assume $J \subseteq \{1, 2, \ldots, n\}$ and $p \in \{1, 2, \ldots, n\} \setminus J$ are such that* (i) *the submatrix B consisting of the columns $A^{(j)}$ $(j \in J)$ is row unisigned and* (ii) *$A^{(p)} = \epsilon(B)$. Then the matrix A' obtained from A by deleting column A^p is also SSC.*

Proof. Since A is SCC, condition (ii) in Theorem 4.3 holds. Consider a signing D given by a $(0, \pm 1)$-vector $s = (s_1, s_2, \ldots, s_m)$. Assume that $s \circ A^{(p)} \ge O$, i.e., the pth column of DA is nonnegative. Let $j \in J$. We show that $s \circ A^{(j)} \ge O$. Since $A^{(p)} = \epsilon(B)$, the column $A^{(j)}$ is obtained from $A^{(p)}$ by replacing some entries (possibly none) by 0. This clearly implies that $s \circ A^{(j)} \ge O$. Next, assume that $s \circ A^{(p)}$ is nonnegative and nonzero. Then there exists an $i \le m$ with $s_i a_{ip} = 1$. Since $A^{(p)} = \epsilon(B)$, there exists a $j \in J$ such that $a_{ij} = a_{ip}$, and therefore $s_i a_{ij} = 1$. So $s \circ A^{(j)}$ is nonnegative (as just shown) and nonzero.

Thus, if A' is obtained by deleting column $A^{(p)}$, we have shown that A' satisfies condition (ii) of Theorem 4.3, so it is SSC. ☐

Note that a submatrix of a row unisigned matrix is also row unisigned. Therefore, Proposition 4.8 gives a number of constraints for a matrix to be minimal SSC.

Example 4.9. Assume that an SSC matrix A has the form

$$A = \left[\begin{array}{rr|rc} 1 & 1 & 1 & \cdots \\ -1 & -1 & -1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ -1 & 0 & -1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \end{array}\right].$$

Then A is not minimal SSC because we can apply Proposition 4.8 with $p = 3$ and $J = \{1, 2\}$, and the matrix obtained by deleting column 3 is also SSC. ☐

5. General results for minimal SSC matrices. This section is on minimal SC and SSC in general. Consider a matrix of the form

$$A = \left[\begin{array}{c|c} A_1 & A_{12} \\ \hline O & A_2 \end{array}\right]. \tag{5.1}$$

If A_1 is SC, then by considering strict signings, we see that A is SC. But if A_1 is SSC, A need not be SSC, for instance,

$$A = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Actually, if A is SSC, then A_2 also must be SSC. However, if in (5.1), both A_1 and A_2 are SSC, then A is SSC independent of A_{12}. (Signings that are nonzero in the rows corresponding to A_1 are covered by the first set of columns, and other signings are covered by the remaining columns.)

Consider A again as in (5.1), where A_1 is SC. Then A is minimal SC if and only if $A = A_1$. Now suppose that A_1 and A_2 are minimal SSC. In particular, A is SSC, and a natural question is if it is also minimal SSC. This, however, is not true in general, as the following matrix shows:

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

This matrix has the form (5.1), where $A_1 = A_2$ is minimal SSC by Corollary 4.6. Let A' be obtained from A by deleting the first column. Then one can check that A' also is SSC. (Signings starting with a 1 are covered by the last four columns.) So A is not minimal SSC, although both A_1 and A_2 have this property. This example shows that the minimal SSC property is more complex than the minimal SC property; even for block triangular matrices it is nontrivial.

The next result gives an upper bound on the number of columns of a minimal SSC matrix. If A is an SSC matrix and D is a signing of order m, we say that the jth column $A^{(j)}$ of A *covers* D whenever $DA^{(j)}$ is nonnegative and nonzero. Let $\mathcal{D}_m$ denote the set of *strict* signings of size m, so $|\mathcal{D}_m| = 2^m$. Recall that E_m denotes the matrix whose columns are all (± 1)-vectors (in some order).

THEOREM 5.1. *Let A be an $m \times n$ $(0, \pm 1)$-matrix which is minimal SSC. Then $n \leq 2^m$. If $n = 2^m$, then A equals* (*up to column permutations*) *the matrix E_m.*

Proof. We prove the result by induction on the number m of rows. The case $m = 1$ is trivial (the row consists of a 1 and a -1), so consider an arbitrary m and assume the result holds for minimal SSC matrices with fewer rows than m.

We shall prove that $n \leq 2^m$. Consider the following property:

(∗) For each $j \leq n$, there is a *strict* signing $D = D^{[j]}$ such that the column $A^{(j)}$ covers $D^{[j]}$ and no other column of A covers $D^{[j]}$.

Observe that if property $(*)$ holds, then $n \leq 2^m$ follows. In fact, one may define a mapping $f : \{1, 2, \ldots, n\} \to \mathcal{D}_m$, where $f(j) = D^{[j]}$, and this mapping is one-to-one. This implies that $n \leq 2^m$.

Thus, it remains to consider the case when property $(*)$ does not hold for a certain j. Since A is minimal SSC, there exists a signing D' (maybe not strict) such that $A^{(j)}$ covers D' and no other column of A covers D'. (Otherwise, the jth column could be deleted.) Let $d' = (d'_1, d'_2, \ldots, d'_m)$ be the diagonal of D'. So, for any $k \neq j$, since $A^{(k)}$ does not cover D', there are two possibilities: either there exists at least one i such that d'_i and a_{ik} are 1 and -1 (in some order), or for all i, $d'_i a_{ik} \geq 0$, and then actually $d'_i a_{ik} = 0$ for each i; otherwise, $A^{(k)}$ would cover D'. The last possibility means that $a_{ik} = 0$ for all i, where $d'_i \neq 0$, so the support of column $A^{(k)}$ is contained in $\{i : d'_i = 0\}$; let K be the set of $k \neq j$ with this property. Define $s = |\{i : d'_i = 0\}|$. So $s \geq 1$. (Otherwise, $A^{(k)} = O$, contradicting minimality.) Permute the rows and columns of A such that rows corresponding to $\{i : d'_i = 0\}$ are placed last and columns corresponding to K are placed last. The permuted A, which we also denote by A for simplicity, is still minimal SSC and has the form

$$A = \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} has $m - s$ rows and A_{22} has s rows. Let D'_1 denote the submatrix of the signing D' that corresponds to the first set of rows in the partition of A, so D'_1 is a strict signing of order $m - s$. Let D_2 be any signing (possibly nonstrict) of order s. Then $\tilde{D} := D'_1 \oplus D_2$ is a signing of order m, and since A is SSC, the matrix $\tilde{D}A$ has a nonnegative nonzero column. But, by definition of the set K above, each column of $\tilde{D}A$ corresponding to a $k \notin K$ contains a negative entry (in the first block row). Therefore, the nonnegative and nonzero column of $\tilde{D}A$ must be among the columns corresponding to K, where we have the zero submatrix in the upper part. It follows that $D_2 A_{22}$ contains a nonnegative and nonzero column. Since D_2 was an arbitrary signing, this shows that the submatrix A_{22} is SSC. Actually, A_{22} is minimal SSC because, if not, we could delete a column of A_{22} and maintain the SSC property, and then we could delete the corresponding column of A and maintain SSC, contradicting the minimality of A. So A_{22} is minimal SSC.

Moreover, A_{11} is minimal SSC. To see this, note that the submatrix consisting of the first $m - s$ rows of A is SSC (a general fact observed before), and zero columns play no role for the SCC property, so A_{11} is SSC. Assume that A_{11} is not minimal SSC. Then some column in this matrix can be deleted and still give an SSC matrix, say, A'_{11}. Let A' denote the submatrix of A obtained by deleting the corresponding column. Then A' is SSC. In fact, any signing having a nonzero among the last s rows is covered by one of the columns corresponding to K (since A_{22} is SSC). Any other signing has only zeros in the last s rows, and it is covered by a column corresponding to A'_{11}, since this matrix is SSC. This proves that A' is SSC, which contradicts that A is minimal SSC. So A_{11} is minimal SSC.

By induction hypothesis, since both A_{11} and A_{22} are minimal SSC, A_{11} has at most 2^{m-s} columns and A_{22} has at most 2^s columns, so

$$n \leq 2^{m-s} + 2^s < 2^m,$$

which proves the desired upper bound.

To prove the last statement, let A be a minimal SSC matrix of size $m \times 2^m$. Then property $(*)$ holds (otherwise, $n < 2^m$), so each of the 2^m columns of A covers a strict signing which no other column covers. But then A cannot contain a zero entry, because that column would cover at least two strict signings, one of them already uniquely covered by another column. So A equals E_m (up to column permutations). ☐

6. Classes of minimal SSC and SC matrices. We now consider a class of minimal SSC matrices of a purely combinatorial nature.

Let

$$A = \left[\begin{array}{c|c} A_1 & -A_1 \end{array}\right], \tag{6.1}$$

where A_1 is an $m \times k$ matrix. If the matrix A_1 is a direct sum, $A_1 = B_1 \oplus B_2$, then

$$A = \left[\begin{array}{cc|cc} B_1 & O & -B_1 & O \\ O & B_2 & O & -B_2 \end{array}\right],$$

which can be column permuted to

$$A = \left[\begin{array}{cc|cc} B_1 & -B_1 & O & O \\ O & O & B_2 & -B_2 \end{array}\right].$$

It follows from the definition of SC and SSC that (i) A is SC if and only if at least one of $[\ B_1 \quad -B_1\ ]$ and $[\ B_2 \quad -B_2\ ]$ is SC and (ii) A is SSC if and only if both of these matrices are SSC.

Let A be of the form (6.1), where A_1 is a $(0,1)$-matrix of size $m \times k$, so A has size $m \times n$, where $n = 2k$. We call such a matrix A *sign-reflective*, or an *SR matrix* for short. In this situation, the SSC property may be reformulated as follows. Let $V = \{1, 2, \ldots, m\}$. Associated with a signing D with diagonal $d_1, d_2, \ldots, d_m$, we define $P_1 = \{i \le m : d_i = 1\}$, $P_2 = \{i \le m : d_i = -1\}$, $P_0 = \{i \le m : d_i = 0\}$. Thus, a signing corresponds to an ordered pair (P_1, P_2), where P_1 and P_2 are disjoint subsets of V with $P_1 \cup P_2 \neq \emptyset$. An SR matrix A corresponds to subsets $S_1, S_2, \ldots, S_k$ of V, where S_j is the support of the jth column of A_1 for $j = 1, 2, \ldots, k$. Define $\mathcal{S}(A) = \{S_1, S_2, \ldots, S_k\}$.

We want to investigate the SSC property for A in terms of properties of the class $\mathcal{S}(A) = \{S_1, S_2, \ldots, S_k\}$. It is natural to introduce a new minimality concept by saying that an SSC matrix A of the form just described is *SR-minimal* SCC if, for each $j \le k$, the deletion of columns j and $k+j$ gives a matrix which is not SSC. The property SR-minimal SC is defined similarly. We use this notion for the remaining part of this section.

With this notation, it follows from Theorem 4.3 that A is SSC if and only if

(6.2) for all disjoint subsets P_1, P_2 of V with $P_1 \cup P_2 \neq \emptyset$, there exists an $S \in \mathcal{S}(A)$ such that S intersects exactly one of the sets P_1 and P_2.

In fact, in this condition, if $S = S_j$, then the corresponding signing D multiplied by the jth column of A_1 is nonzero and either nonnegative or nonpositive. Since the $(k+j)$th column of A is the negative of the jth column of A, the mentioned equivalence follows.

Similarly, from Theorem 4.1, we obtain a characterization of the SC property: A is SC if and only if

(6.3) for every partition P_1, P_2 of V, there exists an $S \in \mathcal{S}(A)$ such that $S \subseteq P_1$ or $S \subseteq P_2$.

Note that (6.2) implies (6.3), but the converse is not true in general.

Let now A be an SR matrix where A_1 is a $(0,1)$-matrix of size $m \times k$ having precisely two ones in every column. Thus, A_1 is the incidence matrix of a graph G with vertices and edges corresponding to rows and columns, respectively. By an *odd near-tree*, we mean a connected graph which contains a unique cycle and where that cycle is odd, i.e., it contains an odd number of edges. Thus, an odd near-tree is obtained from a tree by adding a single edge such that the cycle formed is odd. We remark that odd near-trees also arise in connection with symmetric transportation polytopes; see [4].

In the following theorem, we may without loss of generality assume that the graph G is connected, due to the decomposition result described initially in this section.

THEOREM 6.1. *Let A be an $m \times n$ SR matrix* (*as in* (6.1))*, where A_1 is the incidence matrix of a connected graph G. Then the following holds:*

(i) *If G is bipartite, then A is not SC and therefore not SSC.*
(ii) *If G is not bipartite, then A is SSC and therefore SC.*
(iii) *A is SSC if and only if A is SC.*
(iv) *A is minimal SSC, or minimal SC, if and only if G is an odd near-tree.*
(v) *Each minimal SSC* (*SC*) *matrix* (*of this form*) *has $2m$ columns.*

Proof.

(i) Let $V = \{1, 2, \dots, m\}$ be the vertex set of G. Assume that G is bipartite. Let I_1 and I_2 be the color classes of G, so every edge in G joins a vertex in I_1 and a vertex in I_2. Choose $P_1 = I_1$ and $P_2 = I_2$. Then each edge in G which intersects either P_1 or P_2 also intersects the other, which means that condition (6.3) does not hold, so A is not SC and therefore not SSC.

(ii) Assume that G is not bipartite. Then G contains an odd cycle. Let P_1 and P_2 be disjoint subsets of V with $P_1 \cup P_2 \neq \emptyset$. Define $P_0 = V \setminus (P_1 \cup P_2)$. This gives a partition of V and each vertex lies in precisely one of the three sets P_1, P_2, and P_0.

Claim. There exists an edge uv such that either (i) *$u \in P_1$, $v \in P_1 \cup P_0$ or* (ii) *$u \in P_2$, $v \in P_2 \cup P_0$.*

To prove this, assume that there is no such edge uv as stated in the claim. So, then each edge uv is of one of two types: (I) either both u and v lie in P_0 or (II) $u \in P_1$ and $v \in P_2$ (or vice versa). Moreover, G cannot contain both an edge of type I and an edge of type II, because connectedness would then give an edge joining a vertex in P_0 and a vertex in either P_1 or P_2, contradicting our initial assumption. Thus, there are two possibilities: either G has only edges of type I, or G has only edges of type II. In fact, the first possibility can be ruled out since there is a vertex lying in $P_1 \cup P_2$. So G has only edges of type II. Moreover, by assumption, G contains an odd cycle. Along this cycle, the vertices alternate between P_1 and P_2, but this is impossible as the cycle is odd. This proves the claim.

Finally, the claim shows that condition (6.2) holds, so A is SSC and therefore SC.

(iii) This follows from (i) and (ii).

(iv) Property (iv) follows from property (ii) for the following reason. In a connected graph G which is not bipartite, we may delete edges (corresponding to columns of A) and thereby produce SSC (and SC) matrices with fewer columns (and such that the remaining graph is connected). This edge deletion may continue as long as the remaining graph contains an odd cycle. Thus, we stop precisely when the graph has exactly one cycle and that cycle is odd, which means that we have an odd near-tree.

(v) This is clear, since in a odd near-tree the number of vertices equals the number of edges. ∎

Example 6.2. The following matrix is SR-minimal SSC:

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right].$$

The corresponding graph is an odd near-tree consisting of a cycle of length 3 plus one edge. ∎

For the matrices considered in Theorem 6.1, the properties SC and SSC coincide, and we may construct large families of SR-minimal SSC (or SC) matrices of size $m \times n$ with $n = 2m$. It is also interesting to note that for a given SR matrix A as considered in the theorem, it is easy to check (by a polynomial-time algorithm) whether A is SSC or SR-minimal SSC using the characterization of the theorem.

Consider again an SR matrix A (of the form (6.1)), but now we permit A_1 to be an arbitrary $(0,1)$-matrix. Then the graph G is replaced by a hypergraph H whose edges are the supports of the first p columns. Assume that H has a connected component which is 2-colorable ("bipartite") in the sense that the vertex set V may be partitioned into two sets I_1 and I_2 so that every edge in that component of H intersects both these sets. Then A is not SC; the argument is as in the proof above (in (6.2), let $P_1 = I_1$ and $P_2 = I_2$). Thus, in this situation, a necessary SC condition is that no component of H is 2-colorable. Consider the following example:

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{array}\right].$$

The corresponding hypergraph has edges $\{1,3,4\}$, $\{1,2\}$, and $\{2,3\}$, so H has an odd cycle. But H is still 2-colorable, as we may choose color classes $I_1 = \{1,3\}$ and $I_2 = \{2,4\}$, and therefore A is not SC. Thus, for hypergraphs, 2-colorability and the nonexistence of odd cycles are not equivalent; for more on this, see [3]. A lot of research has been done on 2-colorability of hypergraphs (also called property B), connections to the Lovász local lemma, and related topics.

Based on this discussion, it is natural to ask, for an SR matrix A of the form (6.1), if it is true that A is SSC if and only if each component C of H satisfies $\chi(C) \geq 3$, where $\chi(C)$ is the chromatic number of C. The answer is negative, and a counterexample is given by

$$A_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Let the edges corresponding to the columns be $S_1, S_2, \ldots, S_6$, and let H be the hypergraph, which is connected. The first three edges form an odd cycle, so $\chi(H) \geq 3$ (it is actually 3). From this, we see that A is SC. However, by choosing $P_1 = \{4, 6, 8\}$, $P_2 = \{5, 7, 9\}$, and therefore $P_0 = \{1, 2, 3\}$, (6.2) is violated, so A is not SSC. It seems that the (SR-minimal) SSC property for the hypergraph situation is much more complicated than for graphs.

The following example shows that the properties minimal SC and minimal SSC do not coincide for SR matrices.

Example 6.3. Let $m \geq 1$ and define the SR matrix

$$A_m = \left[\begin{array}{c|c} L_m & -L_m \end{array}\right],$$

where $L_m = [l_{ij}]$ is the $m \times m$ $(0,1)$-matrix with all ones on the diagonal and the upper triangular part, i.e., $l_{ij} = 1$ for $1 \leq i \leq j \leq m$, and $l_{ij} = 0$ otherwise. For instance, for $m = 3$ we have

$$A_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right].$$

Then one can easily check that A_m satisfies (6.2), so A_m is SSC. This also follows from Corollary 4.6. Moreover, in (6.2), if we let $P_1 = \{i\}$ and $P_2 = \{i+1, i+2, \ldots, m\}$, then S_i is the only set in $\mathcal{S}(A_m)$ that satisfies the condition in (6.2). Therefore, A_m is minimal SSC. However, for $m \geq 2$, A_m is not minimal SC, because we can delete any set S_j for some $j \geq 2$, and (6.3) still holds due to the set S_1. □

We now consider a special class of SR matrices. For $1 \leq p \leq m$, define $\mathcal{F}_{p,m}$ as the class of all p-subsets, i.e., subsets of cardinality p, of the set $V = \{1, 2, \ldots, m\}$. So $|\mathcal{F}_{p,m}| = \binom{m}{p}$. Assume that $p \leq \lceil m/2 \rceil$ and let A be such that $\mathcal{S}(A) = \mathcal{F}_{p,m}$. Then A is SSC because, if P_1, P_2 are disjoint subsets of V with $P_1 \cup P_2 \neq \emptyset$, then either $V \setminus P_1$ or $V \setminus P_2$ contains at least $\lceil m/2 \rceil$ elements and therefore that set contains a set in $\mathcal{S}(A)$ which may be chosen to intersect P_1 or P_2, and (6.2) holds.

The next result concerns minimal SC matrices. Define γ_m to be the maximum number n of columns of an SR-minimal $m \times n$ SC matrix A of the form (6.1); where A_1 is a $(0,1)$-matrix. In the following result on γ_m, we distinguish between m odd and even.

THEOREM 6.4. *If A is an SR-minimal SC matrix of the form* (6.1)*, then $\mathcal{S}(A)$ is an antichain, i.e., no set in $\mathcal{S}(A)$ is contained in another. Let p be a positive integer. Then $\gamma_{2p-1} = 2\binom{2p-1}{p}$ and the maximum number γ_{2p-1} is attained for a unique SR-minimal SC matrix A* (*up to column permutations*)*, which is the matrix corresponding to the class $\mathcal{F}_{p,2p-1}$. Moreover, $2\frac{p}{p+1}\binom{2p}{p} \leq \gamma_{2p} < 2\binom{2p}{p}$.*

Proof. To prove that $\mathcal{S}(A)$ is an antichain, let $\mathcal{S}(A) = \{S_1, S_2, \ldots, S_n\}$. Assume $S_i \subseteq S_j$ for distinct i, j. Since A is SC, property (6.3) holds. Consider a partition P_1, P_2 of V. Since $S_i \subseteq S_j$, we have that if $S_j \subseteq P_1$ or $S_j \subseteq P_2$, then also $S_i \subseteq P_1$ or $S_i \subseteq P_2$. But this implies that the two columns corresponding to S_j of A can be deleted and the resulting matrix is still SC, contradicting the SC SR-minimality. This proves that $\mathcal{S}(A)$ is an antichain.

Now, by a theorem of Sperner [1], [13], the largest size of an antichain in a set of m elements is $\binom{m}{\lfloor m/2 \rfloor}$, so

$$\gamma_m \leq 2\binom{m}{\lfloor m/2 \rfloor}.$$

Assume $m = 2p-1$, so m is odd. Let A be such that $\mathcal{S}(A) = \mathcal{F}_{p,m}$. As proved above, A is SC (even SSC). To prove that A is SR-minimal SC, let A' be obtained from A by removing the column whose support is some set S in $\mathcal{F}_{p,2p-1}$ (and the corresponding column, which is the negative of the mentioned one). In condition (6.3), let $P_1 = S$ and $P_2 = S^c$ (the complement). No $S' \in \mathcal{F}_{p,2p-1}$ is contained in S, except S itself. Moreover, no $S' \in \mathcal{F}_{p,2p-1}$ is contained in S^c as $|S^c| = p-1$. Thus, deleting any column of A gives a matrix which is not SC. So A is SR-minimal SC. Since $|\mathcal{F}_{p,2p-1}| = \binom{2p-1}{p}$, we conclude that $\gamma_{2p-1} = 2\binom{2p-1}{p}$ and that this maximum is attained for the mentioned matrix A. The uniqueness also follows from Sperner's theorem since the only antichain, for which A is SR-minimal, attaining the upper bound is the one described above.

Now, consider γ_{2p}, and let A be a matrix such that $\mathcal{S}(A) = \mathcal{F}_{p,2p}$. If we delete a single column, it is easy to check that the resulting matrix is also SC, so $\gamma_{2p} < 2\binom{2p}{p}$. We claim that

$$\gamma_{2p} \geq 2\left(\frac{1}{2}\binom{2p}{p} + \binom{2p-1}{p+1}\right) = 2\frac{p}{p+1}\binom{2p}{p}.$$

In fact, let A be the SR-minimal SC matrix with $2p$ rows with A_1 (as in (6.1)) given by the following: (i) A_1 has as columns the incidence vectors of all the $\frac{1}{2}\binom{2p}{p}$ p-subsets of $\{1,2,\ldots,2p\}$ containing 1 (one set from each partition of $\{1,2,\ldots,2p\}$ into two sets of cardinality p) and (ii) all the $\binom{2p-1}{p+1}$ $(p+1)$-subsets not containing 1. Then this satisfies the partition property in (6.2). Moreover, A is minimal with respect to this property: Suppose that we delete one of the p-sets, say, U, containing 1. Then the partition $U, \overline{U}$ does not satisfy the property (6.2). Alternatively, if we delete one of the $(p+1)$-sets, say, V, not containing 1, then the partition $V, \overline{V}$ does not satisfy the property (6.2). This proves the minimality and therefore the desired lower bound on γ_{2p}. ◻

Note that Example 6.3 shows that there are minimal SSC matrices for which the class $\mathcal{S}(A)$ is not an antichain. However, we have the following result.

Proposition 6.5. *If A is an SR-minimal SSC matrix, then no set in $\mathcal{S}(A)$ is the union of other sets in $\mathcal{S}(A)$.*

Proof. Assume that $S_j \in \mathcal{S}(A)$ is the union of some other sets $S_i \in \mathcal{S}(A)$ $(i \in I)$. Consider condition (6.3) and disjoint sets P_1, P_2 of V with at least one of these sets nonempty. Assume that S_j intersects precisely one of the sets P_1 and P_2. This implies that there exists an $i \in I$ such that S_i intersects precisely one of the sets P_1 and P_2. Consequently, columns j and $j+k$ of A can be deleted, and the resulting matrix is still SSC, contradicting the SC SR-minimality. ◻

For instance, Proposition 6.5 implies that an SR-minimal SSC matrix cannot contain all p-subsets and all r-subsets for some $p \neq r$ (because if $r < p$, a p-subset is the union of some r-subsets).

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