Math 340 (row space, column space, null space of a matrix)

Let $A = [a_{ij}]$ be an m by n matrix. Then

1. the *row space* of A is the subspace of R^n spanned by the rows of A, so all vectors of the form
$$y^T A \text{ where } y \text{ is in } R^m;$$

2. the *column space* of A is the subspace of R^m spanned by the columns of A, so all vectors of the form
$$Ax \text{ where } x \text{ is in } R^n;$$

3. the *null space* of A is the subspace of R^n consisting of the solutions of the homogeneous system $Ax = 0$.

Doing EROs is the same as multiplying on the left by an invertible (i.e., nonsingular matrix) E: the matrix EA is obtained from A by doing EROs on A.

The effect of EROs on an m by n matrix A, that is, the effect on A by multiplying A on the left by an invertible matrix, is thus:

1. ERO's don't change the row space of A, because
$$y^T A = y^T E^{-1} EA = (y^T E^{-1})(EA) = z^T(EA) \text{ where } z^T = y^T E^{-1},$$
but they can change the linear relations among the rows (so that rows that were linearly independent become dependent after EROs and vice-versa).

2. ERO's don't change the linear relations among the columns of A. Thus columns that were linearly independent (or dependent) remain linearly independent (or dependent) after EROs:

   $Ax = 0$ implies $(EA)x = E(Ax) = E0 = 0$ and $(EA)y = 0$ implies, since E is invertible, that $E^{-1}(EA)y = 0$, that is $Ay = 0$.

   EROs will in general change the column space.

3. EROs do not change the null space of the matrix A. In fact as the above shows the vectors in the null space of A are just the linear relations among the columns of A (including the trivial relation).

If by EROs applied to the matrix A we get the (reduced or not) row echelon form U of A. Then the rows of U containing a pivot are linearly independent (this should be clear, look at an example) and span the row space of A (since the row space hasn't changed and the zero rows are of no use). **Thus the rows of U containing a pivot form a basis (!) of the row space of A.**

Also the columns of U containing a pivot are linearly independent (this should be clear, look at an example); thus the corresponding columns of A are linearly independent. Moreover, every other column of U is a linear combination of the columns of U containing a pivot (again look at an example). Thus every other column of U is in the span of the columns containing a pivot. Since the linear relations among the columns don't change under EROs, every column of A is a linear combination of the columns of A corresponding to the columns of U containing a pivot. Since those columns of A are linearly independent, it follows that **the columns of A corresponding to the columns of U containing a pivot form a basis (!) of the column space of A.**

In particular, the dimension of the row space of A is the same as the dimension of the column space of A, both equaling the number of pivots in the (reduced) row echelon form of A. **Wow! What about that!**

A basis of the null space of A also can be obtained from the reduced echelon form U of A by writing the solutions of the homogeneous system $Ax = 0$ (using U of course) in terms of the *free* variables. Since the number of free variables equals n (the total number of variables) minus the number of pivots, the **dimension of the null space of A equals n minus the number of pivots.** Since the dimension of the row space of A equals the dimension of the column space of A equals the number of pivots, we thus obtain

1. The dimension of the row space of A plus the dimension of the null space of A equals n.

2. The dimension of the column space of A plus the dimension of the null space of A equals n.

Wow again!

Anything you don't understand of the above, we can discuss on Tuesday in class. But I urge you to study this before class on Tuesday and look at examples.

RAB, February 16, 2006