Math 240, Spring Semester 2000-01 NAME:
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Final Exam: May 14, 2001 Total Points:

1. Determine a statement that is logically equivalent to

$$p \to (q \to r)$$

that contains no implications $\to$.

2. Determine the truth values of the following statement (True, False, Can't be determined). (The symbol $\neg$ is used for negation.)

- $\neg(\forall x P(x)) \Leftrightarrow \exists P(x)$
- $\forall y \exists x (x^2 + 1 = y^2)$
- $\exists x \forall y (x^2 + 1 = y^2)$
- If $\sqrt{-1}$ is a real number, then the chromatic number of every graph with 5 vertices is 5.
- $(p \wedge (p \to q) \to q$
- $(\neg(p \to q)) \to p$

3. Use mathematical induction to prove

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = (n-1)2^{n+1} \quad \text{for all } n \geq 1.$$

4. Prove using the pigeon-hole principle that no matter which 101 numbers I choose from $\{1, 2, 3, \ldots, 200\}$, there must be two of them, a and b, such that $a \mid b$.

5. (a) Count the number of permutations of the letters in the word INDISCRETE if the vowels must be next to one another.

(b) A movie theatre sells 5 different kinds of candy bars and has on hand a large supply of each kind. If I want to buy 10 bars, determine the number of possibilities for my purchase.

6. (a) In urn I there are are the nine integers $1, 2, \ldots, 9$ and in urn II there are the nine letters A, B, ..., I. In this experiment I reach into

urn I with my left hand and simultaneously grab two numbers and I reach into urn II with my right hand and simultaneously graph two letters. What is the probability that I get two **even integers** and two **vowels**?

(b) Now empty the contents of urn II into urn I. If I reach into the (new) urn I with one hand and grab four objects, what is the probability that I get two **even integers** and two **vowels**?

7. For each of the following relations R on a set A, determine whether it is an *equivalence* relation, *partial order*, or *both* or *neither*.

- $A = \{1, 2, \ldots 150\}$, xRy if and only if $x = y$
- $A = \{1, 2, \ldots, 98\}$, xRy if and only if x and y have the same remainder when divided by 6.
- $A = \{1, 2, \ldots, 100\}$, xRy if and only if $|x - y| \leq 2$
- A equals the vertices of a directed rooted tree, xRy if and only if $x = y$ or there is a directed path from x to y.
- A is the set of people taking this exam, xRy if and only if x and y have the same age and the same hometown.

8. Determine whether **T**rue or **F**alse. If **F**alse, give a counterexample.

- If R and S are equivalence relations, the $R \cap S$ is also an equivalence relation.
- If R and S are equivalence relations, the $R \cup S$ is also an equivalence relation.
- If R and S are partial orders, the $R \cap S$ is also a partial order.
- If R and S are partial orders, the $R \cup S$ is also a partial order.
- If $(A, \leq)$ is a poset, then for all a and b in A, $a \leq b$ or $b \leq a$.
- If $(A, \leq)$ is a lattice, then there is an element m of A such that $a \leq m$ for all a in A.

9. Let a relation R on $\{1, 2, 3, 4\}$ be defined by the matrix

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Apply Warshall's algorithm and determine W_0, W_1, W_2, W_3, W_4.

10. Let functions f, g, h, k be defined on Z^+ by

$$f(n) = n^4+n^2+1 \quad g(n) = n \log n \quad h(n) = n^3 = 1000, \quad k(n) = n^3+1000n^2.$$

Draw the digraph of the relation $R =$ big-O on $\{f, g, h, k\}$ (so an arrow from one function to another means that the first function is big-O of the second).

11. Let the functions $f : Z^+ \to Z^+$ and $g : Z^+ \to R^+$ be defined by:

$$f(n) = 2n \quad g(n) = \sqrt{n}.$$

Verify mentally that f and g are one-to-one and then determine the inverse function $(g \circ f)^{-1}$.

12. Determine the chromatic polynomial of (a triangulation of a cycle of length 5 - to be drawn).

13. Two equivalence relations R and S on the set $A = \{1, 2, \ldots, 10\}$ are defined below by their corresponding partitions of A:

$$R : \quad \{1, 2, 9\}, \{4, 5\}, \{6, 7, 8\}, \{3, 10\}$$

$$S : \quad \{1, 3\}, \{2, 9, 10\}, \{4, 7\}, \{5, 6, 8\}.$$

(a) Determine the partition of A corresponding to the equivalence relation $R \cap S$.

(b) Determine the partition of A corresponding to the equivalence relation $R \vee S$ (i.e. the transitive closure of $R \cup S$.)

14. (a) Construct the network whose maximum flows give the maximal matchings for the relation R determined by

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(b) How many complete matchings are there if

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

15. In the network below we have started one pass through the labeling algorithm applied to the flow whose values are given on the edges (in a pair of numbers c, f on an edge c denotes the capacity of the edge and f denotes the flow value). Complete this pass through the labeling algorithm indicating the new flow and its value. (You should **not** start another pass even if the new flow is not maximal).