Loopy, Hankel, and combinatorially skew-Hankel tournaments

CrossMark

Richard A. Brualdi [a,*], Eliseu Fritscher [b]

[a] Department of Mathematics, University of Wisconsin, Madison, WI 53706, United States
[b] Instituto de Matemática, Universidade Federal Do Rio Grande Do Sul, 91509-900 Porto Alegre-RS, Brazil

A R T I C L E   I N F O

A B S T R A C T

We investigate tournaments with a specified score vector having additional structure: loopy tournaments in which loops are allowed, Hankel tournaments which are tournaments symmetric about the Hankel diagonal (the anti-diagonal), and combinatorially skew-Hankel tournaments which are skew-symmetric about the Hankel diagonal. In each case, we obtain necessary and sufficient conditions for existence, algorithms for construction, and switches which allow one to move from any tournament of its type to any other, always staying within the defined type.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Let K_n be the complete graph with vertices $\{1, 2, \ldots, n\}$. A *tournament* T of order n is a digraph obtained by orienting the edges of K_n. Listing the vertices of T in an arbitrary order, we can form the adjacency matrix of T, which is then called a *tournament matrix*. This is an $n \times n$ (0, 1)-matrix $T = [t_{ij}]$ such that $t_{ii} = 0$ for all i and $t_{ij} = 1 - t_{ji}$ for $i \neq j$, equivalently, such that $T + T^t = J_n - I_n$ where T^t is the transpose of T and J_n is the $n \times n$ matrix of all 1s. In general, we shall not distinguish between a tournament and a corresponding tournament matrix. We refer to both as tournaments and label both as T as we have done above.

We define an $n \times n$ (0, 1)-matrix $A = [a_{ij}]$ to be *combinatorially skew-symmetric* about the main diagonal provided that $a_{ij} = 1 - a_{ji}$ for $i \neq j$. Thus a tournament matrix is combinatorially skew-symmetric about the main diagonal. An $n \times n$ tournament can be considered as the result of a round-robin tournament with players $1, 2, \ldots, n$ where each pair of players compete in a game with player i winning (respectively, losing) the game with player j if and only if $t_{ij} = 1$ (respectively, $t_{ij} = 0$). The *score vector* of T is $R = (r_1, r_2, \ldots, r_n)$ where r_i is the number of wins of player i and so the ith row sum of T. The score vector of T is also the vector of outdegrees of the vertices of T. The score vector R determines the *losing vector* $S = (s_1, s_2, \ldots, s_n)$ where $s_i = (n - 1) - r_i$ is the number of losses of player i and so the ith column sum of T. This is because each player plays $n - 1$ games and the sum of the outdegree and indegree of each vertex is $n - 1$. Changing the order of the vertices replaces T with PTP^t for some permutation matrix P whose score vector is RP^t.

A tournament matrix has 0s on its main diagonal. This is natural from the competition point of view above. Yet, from the matrix point of view there is no reason to insist that only 0s occur on the main diagonal. Thus we shall also consider

---

* Corresponding author.
  E-mail addresses: brualdi@math.wisc.edu (R.A. Brualdi), eliseu.fritscher@ufrgs.br (E. Fritscher).

$n \times n$ tournaments $T = [t_{ij}]$ where each t_{ii} may be 0 or 1. One possible interpretation of this is the following. Before the round-robin competition begins, each player i flips a coin after calling heads or tails. If player i calls the coin correctly, the player gets a point, and we set $t_{ii} = 1$; otherwise, we set $t_{ii} = 0$. A correct call adds 1 to a player's score. The score vector $R = (r_1, r_2, \ldots, r_n)$ is obtained, as above, by counting the number of 1s in each row of T. The losing vector is still defined in terms of the score vector as $S = (s_1, s_2, \ldots, s_n)$ where $s_i = n - r_i$ for all i. But now S is not in general the vector of column sums of T. If $S' = (s'_1, s'_2, \ldots, s'_n)$ is the column sum vector of T, then $r_i + s'_i = n - 1$ or $n + 1$ depending on whether $t_{ii} = 0$ or $t_{ii} = 1$. We call a tournament T obtained in this way a *loopy tournament*, since viewed as a digraph there may be loops at the vertices. The trace of T counts the number of correct calls of the coin flips. Every tournament is also a loopy tournament. The set of all tournaments with score vector R is usually denoted by $\mathcal{T}(R)$. We denote the set of all loopy tournaments with score vector R by $\mathcal{T}^\ell(R)$.

Example 1. The following is a 5×5 loopy tournament:

$$T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{where } T + T^t = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

The score vector of T is $(2, 3, 2, 3, 2)$; the losing vector is $(3, 2, 3, 2, 3)$. The column sum vector of T is $(2, 3, 2, 1, 4)$.

We shall also consider tournaments with additional structure. Tournaments and loopy tournaments are $n \times n(0, 1)$-matrices which are combinatorially skew-symmetric about the main diagonal. The *anti-diagonal* of an $n \times n$ matrix consists of the positions $\{(i, n + 1 - i) : 1 \le i \le n\}$. In a Hankel matrix, the entries in these positions are constant, as are the entries in each of the other $2(n - 1)$ diagonals parallel to the anti-diagonal. We use this association to refer to the anti-diagonal as the *Hankel diagonal*. If $A = [a_{ij}]$ is an $n \times n$ matrix, then we define its *Hankel transpose* to be the matrix $A^h = [a'_{ij}]$ obtained from A by transposing across the Hankel diagonal and thus for which $a'_{ij} = a_{n+1-j, n+1-i}$ for all i and j.[1] It is straightforward to check that, as for ordinary transpose, the Hankel transpose satisfies $(XY)^h = Y^h X^h$ for $n \times n$ matrices X and Y, and that $(X^t)^h = (X^h)^t$, written as $X^{th} = X^{ht}$.

A *Hankel tournament* is defined to be a tournament T for which $T^h = T$. Thus a $(0, 1)$-matrix $T = [t_{ij}]$ is a Hankel tournament if and only if $t_{ii} = 0$ for $i = 1, 2, \ldots, n$, and

$$t_{n+1-j, n+1-i} = t_{ij} = 1 - t_{ji} = 1 - t_{n+1-i, n+1-j} \quad \text{for all } i \ne j.$$

The entries on the Hankel diagonal of a Hankel tournament can be 0 or 1 but, by the combinatorial skew-symmetry of a tournament, there must be $\lfloor \frac{n}{2} \rfloor$ 1s on the Hankel diagonal; if n is odd, then the entry $t_{(n+1)/2, (n+1)/2}$ on the Hankel diagonal equals 0, since T is a tournament. Reordering the columns of a Hankel tournament from last to first (and leaving the rows as is), we obtain a symmetric matrix $\widetilde{T}$ which has $\lfloor \frac{n}{2} \rfloor$ 1s on the main diagonal, and thus $\widetilde{T}$ is the adjacency matrix of a loopy graph of order n with $\lfloor \frac{n}{2} \rfloor$ loops. We call this graph the *Hankel loopy graph* of a Hankel tournament T and denote it by $H(T)$. The set of all Hankel tournaments with score vector R is denoted by $\mathcal{T}_H(R)$.

Example 2. The following is a Hankel tournament T of order 5 and its associated symmetric matrix $\widetilde{T}$:

$$T = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ \hline \end{array} \quad \text{and} \quad \widetilde{T} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ \hline \end{array}.$$

The graph $H(T)$ is a path $1 - 2 - 5 - 3 - 4$ with a loop at vertex 1 and at vertex 4.

We also define a *combinatorially skew-Hankel tournament* to be an $n \times n(0, 1)$-matrix which is combinatorially skew-symmetric about both the main diagonal and the Hankel diagonal, and which has only 0s on both its main diagonal and its Hankel diagonal. Let D_n be the $n \times n(0, 1)$-matrix with 1s on the main diagonal and on the Hankel diagonal and 0s elsewhere. Thus the $n \times n(0, 1)$-matrix $T = [t_{ij}]$ is a combinatorially skew-Hankel tournament if and only if $T^t = J_n - D_n - T$ and $T^h = J_n - D_n - T$. Thus, if T is a combinatorially skew-Hankel tournament, then $T^t = T^h$ or, put another way, $T^{th} = T$. In terms of its entries, T is a combinatorially skew-Hankel tournament if and only if

$$t_{ii} = t_{i, n+1-i} = 0 \quad \text{for all } i,$$
$$t_{ji} = 1 - t_{ij} \quad \text{for all } j \ne i, n + 1 - i,$$
$$t_{n+1-j, n+1-i} = 1 - t_{ij} \quad \text{for all } j \ne i, n + 1 - i.$$

---

[1] By analogy with Toeplitz matrices, it would be natural to call the main diagonal the *Toeplitz diagonal* and to call ordinary transpose the *Toeplitz transpose*.

Thus if T is a combinatorially skew-Hankel tournament, then $t_{ij} = t_{n+1-i,n+1-j}$ for all i and j. Strictly speaking, a combinatorially skew-Hankel tournament is not a tournament because skew-symmetry does not hold for symmetrically opposite elements on the Hankel diagonal. Viewing T with respect to either its main diagonal or its Hankel diagonal, we have a round-robin tournament in which for each i, players i and $n + 1 - i$ do not play a game. The set of all combinatorially skew-Hankel tournaments with a prescribed score vector is denoted by $\mathcal{T}_{H*}(R)$.

Example 3. The following matrix is a 5×5 combinatorially skew-Hankel matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where the four shaded entries determine all the other entries off the main and Hankel diagonals. Note that, in general, a combinatorially skew-Hankel matrix is invariant under a rotation by 180°.

In Section 2 we first extend Landau's theorem for the existence of a tournament with a prescribed score vector by weakening the usual monotonicity assumption. We then use this theorem to obtain necessary and sufficient conditions for the existence of a loopy tournament with a prescribed score vector. We show that this result can also be obtained by associating with an $n \times n$ loopy tournament an $(n + 1) \times (n + 1)$ ordinary tournament and then using Landau's theorem directly. In Section 3 we consider Hankel tournaments and obtain necessary and sufficient conditions for the existence of a Hankel tournament with a prescribed score vector. In Section 4 we consider combinatorially skew-Hankel tournaments and obtain necessary and sufficient conditions for their existence with a prescribed score vector. We also present algorithms for construction of these three types of tournaments.

Let T_1 and T_2 be two tournaments in $\mathcal{T}(R)$ considered as digraphs. It is a basis fact (see e.g. [1,7]) that one may go from T_1 to T_2 by a finite sequence of switches each of which reverses the directions of the edges of a 3-cycle (so resulting in another tournament in $\mathcal{T}(R)$). We identify elementary moves that enable one to go from T_1 to T_2 when (i) $T_1, T_2 \in \mathcal{T}^\ell(R)$, (ii) $T_1, T_2 \in \mathcal{T}_H(R)$, and (iii) $T_1, T_2 \in \mathcal{T}_{H*}(R)$. In each case, the elementary moves always produce another tournament in the same class.

2. Loopy tournaments

One of the most well-known theorems on tournaments is Landau's theorem [3] (see also [1,2,4,6–8]) which asserts that a vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of an $n \times n$ tournament if and only if

$$\sum_{i \in J} r_i \geq \binom{|J|}{2}, \quad (J \subseteq \{1, 2, \ldots, n\}), \text{ with equality if } J = \{1, 2, \ldots, n\}. \tag{1}$$

If T is a tournament with score vector R, then for each permutation matrix P, PTP^t is a tournament with score vector RP^t. Thus without loss of generality, one can assume that R is monotone nondecreasing, that is, $r_1 \leq r_2 \leq \cdots \leq r_n$. With this assumption, (1) is equivalent to

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2}, \quad (k = 1, 2, \ldots, n), \text{ with equality if } k = n. \tag{2}$$

It is usually under the assumption that R is nondecreasing that Landau's theorem is proved. But this monotone assumption can be weakened to provide a somewhat stronger theorem. Let k be a nonnegative integer. The vector R is *k-nearly nondecreasing* provided there is a $(0, 1, \ldots, k)$-vector $u = (u_1, u_2, \ldots, u_n)$ such that $R - u$ is nondecreasing. Thus R is k-nearly nondecreasing if and only if $r_j \geq r_i - k$ for $1 \leq i < j \leq n$. If $k = 0$, then R is nondecreasing. If $k = 1$, then we use *nearly nondecreasing* (as used in [5]) instead of k-nearly nondecreasing. For example, $(3, 2, 3, 4, 3, 4)$ is nearly nondecreasing.

Lemma 4. *Let* $R = (r_1, r_2, \ldots, r_n)$ *be a 2-nearly nondecreasing vector of nonnegative integers. Assume that R satisfies Landau's inequalities* (2). *Then the nondecreasing rearrangement of R also satisfies Landau's inequalities* (2).

Proof. We prove the lemma by induction on the number α of pairs (i, j) such that $i < j$ and $r_i - 2 \leq r_j \leq r_i - 1$. If $\alpha = 0$, then R is nondecreasing and there is nothing to prove. Assume that R is not nondecreasing, and suppose that i and j satisfy $i < j$ and $r_j \leq r_i - 1$. Then for each k with $i < k < j$, $r_k \geq r_i - 2$. Hence there exists p with $i \leq p < j$ such that $r_{p+1} = r_p - l$ where $l = 1$ or 2. We switch r_p and r_{p+1} thereby decreasing α. Suppose that $(\sum_{i=1}^{p-1} r_i) + r_{p+1} < \binom{p}{2}$. Since $\sum_{i=1}^{p} r_i \geq \binom{p}{2}$, we conclude that $\sum_{i=1}^{p} r_i = \binom{p}{2} + h$ where $0 \leq h \leq l - 1$. Since R satisfies Landau's inequalities (2), we calculate that

$$r_{p+1} = \left(\sum_{i=1}^{p+1} r_i \right) - \left(\sum_{i=1}^{p} r_i \right) \geq \binom{p+1}{2} - \left(\binom{p}{2} + h \right) = p - h$$

and

$$r_p = \left(\sum_{i=1}^{p} r_i\right) - \left(\sum_{i=1}^{p-1} r_i\right) \leq \left(\binom{p}{2} + h\right) - \binom{p-1}{2} = p - 1 + h.$$

Thus $p - h + l \leq r_{p+1} + l = r_p \leq p - 1 + h$, and so $l \leq 2h - 1$. Since $h \leq l - 1$, we have $l \geq 3$, a contradiction. We conclude that $(\sum_{i=1}^{p} r_i) + r_{p+1} \geq \binom{p}{2}$, and thus that switching the order of r_p and r_{p+1} produces a vector R' that also satisfies Landau's inequalities. Since α is decreased, the lemma follows by induction. $\quad\square$

Theorem 5. *Let $R = (r_1, r_2, \ldots, r_n)$ be a 2-nearly nondecreasing vector of nonnegative integers. Then R is the score vector of a tournament if and only if* (2) *holds.*

Proof. If R is the score vector of a tournament, then (2) holds. Now assume that (2) holds. By Lemma 4 the nondecreasing rearrangement R' of R satisfies (2). Hence by Landau's theorem, there is a tournament T with score vector R'. There is an $n \times n$ permutation matrix P such that $R' = RP^t$. Then PTP^t is a tournament with score vector R. $\quad\square$

Let $T = [t_{ij}]$ be a loopy tournament with score vector $R = (r_1, r_2, \ldots, r_n)$. Since simultaneously permuting the rows and columns of T results in another loopy tournament (with the same number of loops), we assume without loss of generality that R satisfies $r_1 \leq r_2 \leq \cdots \leq r_n$. The number of 1s on the main diagonal of T is some integer, denoted as $n - t$, where $0 \leq t \leq n$. The sum of all the entries of T equals $\binom{n}{2} + (n - t)$. Landau's theorem for the existence of a tournament with a prescribed score vector can be used to determine the existence of a loopy tournament with a prescribed score vector. If a is an integer, then $a^+ = \max\{0, a\}$.

Theorem 6. *Let $R = (r_1, r_2, \ldots, r_n)$ be a vector of nonnegative integers with $r_1 \leq r_2 \leq \cdots \leq r_n$. Then there exists a loopy tournament with score vector R if and only if there is an integer t with $0 \leq t \leq n$ such that*

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2} + (k - t)^+, \quad (k = 1, 2, \ldots, n), \ \text{with equality when } k = n. \tag{3}$$

When these conditions are satisfied, the number of 1 s on the main diagonal of the loopy tournament is $n - t$ and these can be taken to be in the last $(n - t)$ positions on the main diagonal.

Proof. The condition (3) is clearly necessary for the existence of a loopy tournament with score vector R and $n - t$ loops. Now suppose that (3) holds. We show that there is a loopy tournament with score vector R for which the $(n - t)$ 1s on the main diagonal occur in those positions corresponding to the $(n - t)$ largest r_i. Let $R' = (r_1', r_2', \ldots, r_n')$ be obtained from R by subtracting 1 from $r_{t+1}, r_{t+2}, \ldots, r_n$. Then R' is nearly nondecreasing and (3) implies that $\sum_{i=1}^{k} r_i' \geq \binom{k}{2}$ for $1 \leq k \leq n$ with equality for $k = n$. Hence by Theorem 5 there exists a tournament T' with score vector R'. Replacing the 0s with 1s in the last $(n - t)$ positions on the main diagonal of T' gives a loopy tournament with score vector R. $\quad\square$

Let R be a nearly nondecreasing vector of nonnegative integers satisfying (2). By Theorem 5 there exists a tournament T with score vector R. Simultaneously permuting the rows and columns of T gives a tournament with score vector R' obtained by applying the same permutation to R. Thus to find T we can first permute R to get a nondecreasing R', apply a known algorithm (for instance, the algorithm of Ryser [7,1]) to obtain a tournament T' with score vector R', and then simultaneously permute the rows and columns of T' to obtain T with score vector R.

To construct a loopy tournament with nondecreasing score vector R satisfying (3), we can first subtract 1 from the largest $n - t$ components of R, and this results in a nearly nondecreasing vector R' which by Theorem 5 satisfies (2). Then we can construct a tournament T' with score vector R' as just outlined above. Replacing the 0s in the last $(n - t)$ positions on the main diagonal with 1s results in a loopy tournament with score vector R.

The proof of Theorem 6 uses the strengthened form of Landau's theorem given in Theorem 5 in order to show existence of a loopy tournament with a prescribed nondecreasing score vector. For a given nondecreasing vector R of nonnegative integers, we now establish a bijection between $\mathcal{T}^{\ell}(R)$ and $\mathcal{T}(R')$ for a certain vector R'. This enables us to identify basic switches, a sequence of which allows us to go from a $T_1 \in \mathcal{T}^{\ell}(R)$ to a $T_2 \in \mathcal{T}^{\ell}(R)$ with all intermediate matrices also in $\mathcal{T}^{\ell}(R)$.

Theorem 7. *Let $R = (r_1, r_2, \ldots, r_n)$ be a nondecreasing vector of nonnegative integers such that there is an integer t with $0 \leq t \leq n$ such that*

$$\sum_{i=1}^{n} r_i = \binom{n}{2} + (n - t).$$

Let $R' = (t, r_1, r_2, \ldots, r_n)$. Then there is a bijection between $\mathcal{T}^{\ell}(R)$ and $\mathcal{T}(R')$. In particular, $\mathcal{T}^{\ell}(R) \neq \emptyset$ if and only if $\mathcal{T}(R') \neq \emptyset$. Moreover, these sets are nonempty if and only if (3) *holds.*

Proof. Let $T = [t_{ij}]$ be an $n \times n$ loopy tournament in $\mathcal{T}^\ell(R)$, and let an $(n + 1) \times (n + 1)$ matrix $T' = [t'_{ij}]$ with rows and columns indexed by $0, 1, 2, \ldots, n$ be defined by

$$
t'_{ij} = \begin{cases}
t_{ij}, & \text{if } 1 \leq i,\, j \leq n \text{ and } i \neq j, \\
t_{ii}, & \text{if } j = 0 \text{ and } 1 \leq i \leq n, \\
1 - t_{jj}, & \text{if } i = 0 \text{ and } 1 \leq j \leq n, \\
0, & \text{if } 0 \leq i = j \leq n.
\end{cases}
$$

Thus T' is obtained from T by horizontally moving the entries on the main diagonal to column 0, vertically moving 1 minus the entries on the main diagonal to row 0, and putting 0s everywhere on the main diagonal. Since T has $(n - t)$ 1s on its main diagonal, T' is a tournament in $\mathcal{T}(R')$. This mapping is reversible and hence there is a bijection between $\mathcal{T}^\ell(R)$ and $\mathcal{T}(R')$.

As remarked in the proof of Theorem 6, the inequalities (3) are necessary for $\mathcal{T}^l(R)$ and hence $\mathcal{T}(R')$ to be nonempty. Now assume that (3) holds. Let $R'' = (r''_1, r''_2, \ldots, r''_n, r''_{n+1})$ be obtained from R' by reordering its entries to be nondecreasing. Then $R'' = (r_1, \ldots, r_p, t, r_{p+1}, \ldots, r_n)$ where $r_p \leq t \leq r_{p+1}$. We have

$$
\sum_{i=1}^{k} r''_i = \begin{cases}
\displaystyle\sum_{i=1}^{k} r_i \geq \binom{k}{2} + (k - t)^+ \geq \binom{k}{2}, & \text{if } k \leq p, \\[2ex]
\displaystyle\left(\sum_{i=1}^{k} r_i \right) + t \geq \binom{k}{2} + t + (k - t)^+, & \text{if } p + 1 \leq k \leq n + 1.
\end{cases}
$$

Since $t + (k - t)^+ \geq k$, we have that

$$
\binom{k}{2} + t + (k - t)^+ \geq \binom{k}{2} + k = \binom{k + 1}{2}.
$$

Thus by Landau's theorem, $\mathcal{T}(R'')$ is nonempty, and hence $\mathcal{T}(R')$ and $\mathcal{T}^\ell(R)$ are nonempty. $\quad\square$

To construct a loopy tournament in $\mathcal{T}^\ell(R)$, we can use any algorithm to construct a tournament in $\mathcal{T}(R')$, and then use the bijection in Theorem 7.

As remarked in the introduction, given any two tournaments T'_1 and T'_2 with the same score vector R', then by reversing 3-cycles we can get from T'_1 to T'_2 where all intermediate matrices are tournaments with score vector R'. The operation of reversing a 3-cycle $i \to j \to k \to i$ in terms of matrices is that of switching the 3×3 matrix $T'[i, j, k]$ determined by rows and columns $i, j,$ and k of T' as shown below:

	i	j	k
i	x	1	0
j	0	y	1
k	1	0	z

$\rightarrow$

	i	j	k
i	x	0	1
j	1	y	0
k	0	1	z

$\qquad\qquad$ (4)

We call this a 3-*cycle switch* and denote it by $\triangle_{i,j,k}$. A 3-cycle switch is reversible and its inverse is $\triangle_{k,j,i}$.

We now take R' and R as given in the proof of Theorem 7, and using the bijection between $\mathcal{T}^\ell(R)$ and $\mathcal{T}(R')$ given there, we identify the switches in $\mathcal{T}^\ell(R)$ corresponding to the 3-cycle switches in $\mathcal{T}(R')$. There are two possibilities: the switch in $\mathcal{T}(R')$ is in rows and columns (a) $\{i, j, k\} \subseteq \{1, 2, \ldots, n\}$ or (b) $\{0, i, j\}$ where $\{i, j\} \subset \{1, 2, \ldots, n\}$.

If (a), then the switches for $\mathcal{T}(R')$ and $\mathcal{T}^\ell(R)$ are the same and hence $\triangle_{i,j,k}$ is an operation that maintains the tournament in the class $\mathcal{T}^\ell(R)$.

If (b), then the switch $\triangle_{0,i,j}$ for $\mathcal{T}(R')$ is

	0	i	j
0	0	1	0
i	0	0	1
j	1	0	0

$\rightarrow$

	0	i	j
0	0	0	1
i	1	0	0
j	0	1	0

,

which, by moving the 1 in column 0 of the left matrix to column j and moving the 1 in column 0 of the right matrix to column i, becomes

	i	j
i	0	1
j	0	1

$\rightarrow$

	i	j
i	1	0
j	1	0

.

In terms of the digraph, the operation (b) reverses the direction of an edge from a non-loop vertex i to a loop-vertex j and moves the loop from j to i. We call this operation an *edge-loop switch* and denote it by $\to \circ_{ij}$. This operation is also reversible and the inverse of $\to \circ_{ij}$ is $\to \circ_{ji}$. Thus we have the following theorem.

Theorem 8. *Let T_1 and T_2 be two loopy tournaments in $\mathcal{T}^\ell(R)$. Then T_1 can be brought to T_2 by a sequence of 3-cycle switches and edge-loop switches where each switch produces a loopy tournament in $\mathcal{T}^\ell(R)$.*

To conclude this section, we remark that, any $n \times n$ tournament can always be collapsed to an $(n-1) \times (n-1)$ loopy tournament using any row and column i with the 1s in column i moved horizontally to the positions on the main diagonal, and then deleting row and column i.

3. Hankel tournaments

We first characterize the score vectors of Hankel tournaments. Let T be a Hankel tournament, and let i be an integer with $1 \le i \le n$. Since T is a tournament, row i of T determines column i, and since T is a Hankel tournament, it also determines row and column $n + 1 - i$. We illustrate this important property in the next example.

Example 9. Let T be an 8×8 Hankel tournament with row 3 given. Then the entries in column 3, and row and column $8 + 1 - 3 = 6$ are determined as shown in

$$
\left[
\begin{array}{cc|c|cc|c|cc}
0 & & 1-a & & & g & & \\
& 0 & 1-b & & & f & & \\
a & b & 0 & c & d & e & f & g \\
& & 1-c & 0 & & d & & \\
& & 1-d & & 0 & c & & \\
1-g & 1-f & 1-e & 1-d & 1-c & 0 & 1-b & 1-a \\
& & 1-f & & & b & 0 & \\
& & 1-g & & & a & & 0 \\
\end{array}
\right],
$$

where the 0s on the main diagonal have been inserted and the Hankel diagonal has been shaded.

Let $R = (r_1, r_2, \ldots, r_n)$ be the score vector of T, and let $S = (s_1, s_2, \ldots, s_n)$ be the column sum vector of T. We have $\sum_{i=1}^{n} r_i = \binom{n}{2}$. Since T is symmetric about the Hankel diagonal, $r_i = s_{n+1-i}$. Since T is a tournament, $r_{n+1-i} = (n-1) - s_{n+1-i} = (n-1) - r_i$ and thus the score vector R satisfies the *Hankel property*

$$r_i + r_{n+1-i} = n - 1 \quad \text{for all } i = 1, 2, \ldots, n. \tag{5}$$

Thus, if n is odd,

$$r_{(n+1)/2} = \frac{n-1}{2}.$$

We next show that there is no loss of generality in assuming that R is nondecreasing.

Lemma 10. *Let $R = (r_1, r_2, \ldots, r_n)$ be a vector of nonnegative integers, and let $R' = (r'_1, r'_2, \ldots, r'_n)$ be obtained from R by rearranging its entries so that $r'_1 \le r'_2 \le \cdots \le r'_n$. Then there exists a Hankel tournament T with score vector R if and only if there exists a Hankel tournament T' with score vector R'. Moreover, T' is obtained by simultaneously permuting the rows and columns of T.*

Proof. Let T be a Hankel tournament with score vector R. We first note that if for some i, we interchange rows i and $(n+1-i)$ and simultaneously interchange columns i and $(n+1-i)$ of T, then the resulting matrix is a Hankel tournament with score vector obtained from R by interchanging r_i and r_{n+1-i}. Thus using a sequence of pairwise interchanges of this sort, we may assume that $r_i \le r_{n+1-i}$ for $1 \le i \le \lfloor n/2 \rfloor$. Since $r_i + r_{n+1-i} = n - 1$, we have

$$r_i \le \frac{n-1}{2} \le r_{n+1-i}, \quad \text{where, if } n \text{ is odd, } r_{(n+1)/2} = \frac{n-1}{2}.$$

If for some i and j with $1 \le i < j \le \lfloor n/2 \rfloor$, we have $r_i > r_j$, then we also have that $\lceil n/2 \rceil \le n + 1 - j < n + 1 - i \le n$ and $r_{n+1-j} > r_{n+1-i}$. Thus we may interchange rows i and j and columns i and j, and also interchange rows $(n+1-i)$ and $(n+1-j)$ and columns $(n+1-i)$ and $(n+1-j)$, and obtain a Hankel tournament with row sum vector obtained from R by interchanging r_i and r_j and by interchanging r_{n+1-i} and r_{n+1-j}. Thus by a sequence of double pairwise interchanges of this sort, we are able to get a Hankel tournament whose row sum vector is a monotone rearrangement of R. The converse follows in a similar way. □

Because of Lemma 10, for the existence of Hankel tournaments with score vector R, it suffices to assume that R is nondecreasing. We show that the only condition needed in addition to Landau's inequalities is that R satisfy the Hankel property.

Theorem 11. *Let $R = (r_1, r_2, \ldots, r_n)$ be a vector of nonnegative integers with $r_1 \leq r_2 \leq \cdots \leq r_n$. Then there exists a Hankel tournament with score vector R if and only if*

$$r_i + r_{n+1-i} = n - 1, \quad (i = 1, 2, \ldots, n) \tag{6}$$

and

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2}, \quad (k = 1, 2, \ldots, n), \text{ with equality if } k = n. \tag{7}$$

Proof. We know that (6) and (7) are necessary for the existence of a Hankel tournament with score vector R. Suppose, to the contrary, that the converse is false. Then clearly $n \geq 3$. We then choose an $R = (r_1, r_2, \ldots, r_n)$ satisfying (6) and (7) for which a Hankel tournament with score vector R does not exist, where n is minimum and for this n, r_1 is minimum. We consider two cases.

Case 1: For each k with $1 \leq k \leq n - 1$, we have strict inequality in (7).

In this case we have $r_1 \geq 1$ and we consider $R' = (r_1', r_2', \ldots, r_n') = (r_1 - 1, r_2, \ldots, r_{n-1}, r_n + 1)$. Then with r_i' replacing r_i, R' satisfies the corresponding conditions (6) and (7). Thus by the minimality condition on r_1, there exists a Hankel tournament $T = [t_{ij}]$ with score vector R'. Since $r_n' - r_1' = (r_n + 1) - (r_1 - 1) = (r_n - r_1) + 2 \geq 2$, there exists an integer p with $2 \leq p \leq n - 1$, such that $t_{1p} = 0$ and $t_{np} = 1$. Since T is a tournament, we have $t_{p1} = 1$ and $t_{pn} = 0$. Since T is a Hankel tournament, we also have $t_{n+1-p,n} = 0$, $t_{n+1-p,1} = 1$, $t_{n,n+1-p} = 1$, and $t_{1,n+1-p} = 0$. Thus with $q = n + 1 - p$, we have the structure:

	1	···	p	···	q	···	n
1			0		0		
⋮							
p	1		0		a		0
⋮							
q	1		1 − a		0		0
⋮							
n			1		1		

where a, on the Hankel diagonal, is 0 or 1. If $a = 1$, then replacing as shown below:

	1	···	p	···	q	···	n
1			0		0		
⋮							
p	1		0		1		0
⋮							
q	1		0		0		0
⋮							
n			1		1		

$\longrightarrow$

	1	···	p	···	q	···	n
1			0		1		
⋮							
p	1		0		0		1
⋮							
q	0		1		0		0
⋮							
n			0		1		

,

we obtain a Hankel tournament with score vector R. If $a = 0$, then replacing as shown below:

	1	···	p	···	q	···	n
1			0		0		
⋮							
p	1		0		0		0
⋮							
q	1		1		0		0
⋮							
n			1		1		

$\longrightarrow$

	1	···	p	···	q	···	n
1			1		0		
⋮							
p	0		0		1		0
⋮							
q	1		0		0		1
⋮							
n			1		0		

,

we obtain a Hankel tournament with score vector R. Both possibilities give a contradiction. Note that if $p = q$, then their common value is $(n + 1)/2$ and the number of replacements above is only four.

Case 2: There is a k with $1 \leq k \leq n - 1$ such that we have equality in (7), that is, $\sum_{i=1}^{k} r_i = \binom{k}{2}$.

We calculate that

$$\sum_{i=1}^{n-k} r_i = \binom{n}{2} - \sum_{i=n-k+1}^{n} r_i = \binom{n}{2} - \sum_{i=1}^{k} (n - 1 - r_i)$$

$$= \binom{n}{2} - k(n-1) + \binom{k}{2} = \binom{n-k}{2}.$$

Hence we may assume that $k \leq n/2$, and we also have that $\sum_{i=1}^{n-k} r_i = \binom{n-k}{2}$.

We now calculate that

$$r_{k+1} = \sum_{i=1}^{k+1} r_i - \sum_{i=1}^{k} r_i = \sum_{i=1}^{k+1} r_i - \binom{k}{2}$$

$$\geq \binom{k+1}{2} - \binom{k}{2} = k.$$

Thus by the monotonicity assumption, $r_i \geq k$ for $k + 1 \leq i \leq n$. Similarly, $r_k \leq k - 1$.

It follows from Landau's theorem that there exists a tournament T_1 with score vector $R_1 = (r_1, r_2, \ldots, r_k)$. Since $r_i + r_{n+1-i} = n - 1$ for all i and $r_k \leq k - 1$, we have $r_{n-k+1} \geq (n-1) - (k-1) = n - k$. The monotonicity assumption on R now implies that $R_2 = (r_{n-k+1} - (n-k), r_{n-k+2} - (n-k), \ldots, r_n - (n-k))$ is a vector of nonnegative integers. Since for $1 \leq i \leq k$, we have

$$r_i + (r_{n+1-i} - (n-k)) = r_i + r_{n+1-i} - (n-k) = (n-1) - (n-k) = k - 1,$$

then $s_i = r_{n+1-i} - (n-k)$ is the ith column sum of T_1. Hence the row sum vector of T_1^h equals R_2.

To summarize thus far, the matrix

$$T = \left[\begin{array}{c|c|c} T_1 & O_{k,n-2k} & O_{k,k} \\ \hline J_{n-2k,k} & X & O_{n-2k,k} \\ \hline J_{k,k} & J_{k,n-2k} & T_1^h \end{array} \right]$$

will be a Hankel tournament with score vector R, provided we can choose X as an $(n-2k) \times (n-2k)$ Hankel tournament with score vector $(r_{k+1} - k, r_{k+2} - k, \ldots, r_{n-k} - k)$, $r_{k+1} - k \leq r_{k+2} - k \leq \ldots \leq r_{n-k} - k$ which we know is nonnegative. We calculate that for $1 \leq l \leq n - 2k$,

$$\sum_{j=1}^{l} (r_{k+j} - k) = \left(\sum_{i=1}^{k+l} r_i - \sum_{i=1}^{k} r_i \right) - lk = \sum_{i=1}^{k+l} r_i - \binom{k}{2} - lk$$

$$\geq \binom{k+l}{2} - \binom{k}{2} - lk = \binom{l}{2}.$$

Since by assumption $r_{k+j} + r_{n+1-(k+j)} = n - 1$, we have

$$(r_{k+j} - k) + (r_{n+1-(k+j)} - k) = (n - 2k) - 1.$$

The minimality assumption now implies that the Hankel tournament X exists, and hence we have a Hankel tournament with score vector R. This contradiction completes this case, and thus the proof of the theorem is complete. □

As a corollary we show that the nondecreasing assumption in Theorem 11 can be weakened to 2-nearly nondecreasing.

Corollary 12. *Let* $R = (r_1, r_2, \ldots, r_n)$ *be a 2-nearly nondecreasing vector of nonnegative integers. Then there exists a Hankel tournament with score vector R if and only if*

$$r_i + r_{n+1-i} = n - 1, \quad (i = 1, 2, \ldots, n) \tag{8}$$

and

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2}, \quad (k = 1, 2, \ldots, n), \text{ with equality if } k = n. \tag{9}$$

Proof. As before the conditions (8) and (9) are necessary for a Hankel tournament with score vector R. Now assume that (8) and (9) hold. By Lemma 4 the nondecreasing arrangement of R also satisfies (9). Thus we need only check that the nondecreasing rearrangement of R also satisfies (8). Then the corollary follows from Theorem 11.

Let $(x_1, x_2, \ldots, x_n)$ be the nondecreasing rearrangement of R. Then for each k, x_k is the kth smallest of $r_1, r_2, \ldots, r_n$ and x_{n+1-k} is the kth largest. Suppose that $x_k = r_j$. It follows from (8) that r_{n+1-j} is the kth largest of $r_1, r_2, \ldots, r_n$ and hence $x_{n+1-k} = n + 1 - r_j$. Therefore $x_k + x_{n+1-k} = r_j + r_{n+1-j} = n - 1$.   □

We now provide an algorithm to construct a Hankel tournament with score vector R. This algorithm is a variant of Ryser's algorithm [7] to determine a tournament with a prescribed score vector (see also pages 220–222 of [1]). The last column of such a tournament determines the last row by the combinatorial skew symmetry property of a tournament and thus, by the symmetry property of a Hankel tournament, determines the first row and first column.

Hankel Algorithm for a $T = [t_{ij}] \in \mathcal{T}_H(R)$ with R nondecreasing

Let $R = (r_1, r_2, \ldots, r_n)$ be a nondecreasing vector of nonnegative integers satisfying the Hankel property $r_i + r_{n+1-i} = n - 1$ for $i = 1, 2, \ldots, n$ and the inequalities $\sum_{i=1}^{k} r_i \geq \binom{k}{2}$ with equality for $k = n$ of (7). We let $S_n = (s_1, s_2, \ldots, s_n)$ where $s_i = n - 1 - r_i$ for $i = 1, 2, \ldots, n$.

Remarks: S_n is the column sum vector of a Hankel tournament with score vector R. If n is odd, then we know that $r_{(n+1)/2} = (n-1)/2$, $r_n \geq (n-1)/2$, and $s_n \leq (n-1)/2$. If n is even, $r_n \geq n/2$ and $s_n \leq (n-2)/2$.

(1) If n is even:

  (a) Let $v = (v_1, v_2, \ldots, v_n)$ be the vector $(\underbrace{0, \ldots, 0}_{n-r_1-1}, \underbrace{1, \ldots, 1}_{r_1}, 0)$, and let $R'_{n-2} = (r'_1, r'_2, \ldots, r'_{n-2})$ be the 2-nearly nondecreasing vector with the Hankel property defined by

$$r'_i = r_{i+1} - v_{i+1} - (1 - v_{n-i}) \quad \text{for } 1 \leq i \leq n - 2.$$

  (b) Let Q_{n-2} be the permutation matrix of the type used in the proof of Lemma 10 such that $R'_{n-2} Q^t_{n-2}$ is a nondecreasing sequence R_{n-2} with the Hankel property.

  (c) Let T_{n-2} be the $(n-2) \times (n-2)$ Hankel tournament obtained by this algorithm applied to the score vector R_{n-2}.

  (d) Let T_n be the Hankel tournament with score vector R defined by

$$T_n = \left[\begin{array}{c|c|c} 0 & v_{n-1} \cdots v_2 & v_1 \\ \hline 1 - v_{n-1} & & v_2 \\ \vdots & Q^t_{n-2} T_{n-2} Q_{n-2} & \vdots \\ 1 - v_2 & & v_{n-1} \\ \hline 1 - v_1 & 1 - v_2 \cdots 1 - v_{n-1} & 0 \end{array}\right].$$

(2) If n is odd:

  (a) Let $R'_{n-1} = (r'_1, r'_2, \ldots, r'_{n-1})$ be the nearly nondecreasing vector defined by

$$r'_i = \begin{cases} r_i, & \text{if } 1 \leq i \leq \dfrac{n-1}{2}, \\ r_{i+1} - 1, & \text{if } \dfrac{n+1}{2} \leq i \leq n - 1. \end{cases}$$

  (b) Let Q_{n-1} be the permutation matrix of the type used in Lemma 10 such that $R'_{n-1} Q^t_{n-1}$ is a nondecreasing vector R_{n-1}.

  (c) Let T_{n-1} be the $(n-1) \times (n-1)$ Hankel tournament obtained by this algorithm applied to the score vector R_{n-1}, and let T'_{n-1} be the Hankel tournament with score vector R'_{n-1} obtained by applying the permutation matrix Q_{n-1}:

$$T'_{n-1} = Q^t_{n-1} T_{n-1} Q_{n-1} = \left[\begin{array}{c|c} A & B \\ \hline J_{(n-1)/2} - B^t & A^h \end{array}\right].$$

  (d) Let T_n be the Hankel tournament with score vector R defined by

$$T_n = \left[\begin{array}{cccc|c|cccc} & & & & 0 & & & & \\ & & A & & \vdots & & B & & \\ & & & & 0 & & & & \\ \hline 1 & \cdots & & 1 & 0 & 0 & \cdots & & 0 \\ \hline & & & & 1 & & & & \\ & J_{(n-1)/2} - B^t & & & \vdots & & A^h & & \\ & & & & 1 & & & & \end{array}\right].$$

(3) Output T_n.

Example 13. Let $n = 7$ and let $R = (1, 2, 2, 3, 4, 4, 5)$. Applying the Hankel algorithm, we obtain the following Hankel tournament in $\mathcal{T}_H(R)$:

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix} .
$$

In carrying out the algorithm, the resulting score vectors are illustrated below where $\to_\pi$ means a permutation is used and $\to_a$ means the resulting score vector after a step of the algorithm:

$$(1, 2, 2, 3, 4, 4, 5) \to_a (1, 2, 2, 3, 3, 4) \to_a (2, 1, 2, 1) \to_\pi (1, 1, 2, 2) \to_a (1, 0) \to_\pi (0, 1).$$

Theorem 14. *The Hankel algorithm constructs a Hankel tournament in $\mathcal{T}_H(R)$ when R satisfies the given conditions.*

Proof. We first assume that n is even. The entries of R' satisfy:

$$
r_i' = \begin{cases}
r_{i+1}, & \text{if } 1 \leq i \leq r_1, \\
r_{i+1} - 1, & \text{if } r_1 + 1 \leq i \leq n - r_1 - 2, \\
r_{i+1} - 2, & \text{if } n - r_1 - 1 \leq i \leq n - 2.
\end{cases}
$$

Since R satisfies the Hankel property, so does R'. Thus to verify that the algorithm gives a Hankel tournament with score vector R, by Corollary 12 we need only verify that R'_{n-2} is 2-nearly nondecreasing and satisfies the corresponding inequalities of (7). Since $r_i' \leq r_{i+1}$ and $r_j' \geq r_j - 2$, we have $r_i' \leq r_{i+1} \leq r_j \leq r_j' + 2$ for $i < j$, and hence R'_{n-2} is 2-nearly nondecreasing.

We now verify that R'_{n-2} satisfies the corresponding inequalities of (7). Again we consider three cases,

Case $1 \leq k \leq r_1$: We have

$$\sum_{i=1}^{k} r_i' = \sum_{i=1}^{k} r_{i+1} \geq \sum_{i=1}^{k} r_i \geq \binom{k}{2}.$$

Case $r_1 + 1 \leq k \leq n - r_1 - 2$: We calculate that

$$
\begin{aligned}
\sum_{i=1}^{k} r_i' &= \sum_{i=1}^{r_1} r_{i+1} + \sum_{i=r_1+1}^{k} (r_{i+1} - 1) \\
&= \left(\sum_{i=1}^{k} r_{i+1} \right) - (k - r_1) = \left(\sum_{i=1}^{k+1} r_i \right) - k \\
&\geq \binom{k+1}{2} - k = \binom{k}{2}.
\end{aligned}
$$

Case $n - r_1 - 1 \leq k \leq n - 2$: We first observe that for $n - r_1 \leq l \leq n - 1$, we have the stronger inequality

$$\sum_{i=1}^{l} r_i \geq \binom{l}{2} + (l - n + r_1 + 1)$$

by considering the minimum number of 1s in the $l \times (l + 1)$ submatrix determined by rows $1, 2, \ldots, l$ and columns $1, 2, \ldots, l, n$. Using this inequality we calculate that

$$
\begin{aligned}
\sum_{i=1}^{k} r_i' &= \sum_{i=1}^{r_1} r_{i+1} + \sum_{i=r_1+1}^{n-r_1-2} (r_{i+1} - 1) + \sum_{i=n-r_1-1}^{k} (r_{i+1} - 2) \\
&= \left(\sum_{i=1}^{k} r_{i+1} \right) - (n - 2r_1 - 2) - 2(k - n + r_1 + 2) \\
&= \left(\sum_{i=1}^{k+1} r_i \right) - (2k + 2 + r_1 - n)
\end{aligned}
$$

$$\geq \binom{k+1}{2} + (k - n + r_1 + 2) - (2k + 2 + r_1 - n)$$

$$= \binom{k+1}{2} - k = \binom{k}{2}.$$

Now we assume that n is odd.

The entries of the vector R'_{n-1} are given by

$$r'_i = \begin{cases} r_i, & \text{if } 1 \leq i \leq \dfrac{n-1}{2}, \\ r_{i+1} - 1, & \text{if } \dfrac{n+1}{2} \leq i \leq n - 1. \end{cases}$$

It follows by inspection that the vector R'_{n-1} is nearly nondecreasing, and we have $r'_i + r'_{n-i} = (n - 1) - 1$ for each i. Now we verify that Landau's inequalities $\sum_{i=1}^{k} r'_i \geq \binom{k}{2}$ hold for $1 \leq k \leq n - 1$ with equality if $k = n - 1$. If $1 \leq k \leq (n-1)/2$, then this is clear. Suppose that $(n-1)/2 \leq k \leq n - 1$. Then

$$\sum_{i=1}^{k} r'_i = \left(\sum_{i=1}^{(n-1)/2} r_i \right) + \left(\sum_{i=(n+3)/2}^{k+1} r_i \right) - \left(k + 1 - \frac{n+3}{2} + 1 \right)$$

$$= \left(\sum_{i=1}^{k+1} r_i \right) - k + \frac{n-1}{2} - r_{(n+1)/2}$$

$$= \left(\sum_{i=1}^{k+1} r_i \right) - k \geq \binom{k+1}{2} - k = \binom{k}{2},$$

with equality if $k = n - 1$. Since R' is nearly nondecreasing, after rearranging its terms to form a nondecreasing vector, we can apply the algorithm for the even case of $n - 1$ and as indicated in the algorithm construct a Hankel tournament with score vector R.   □

Let T^* be the Hankel tournament in $\mathcal{T}_H(R)$ constructed by the Hankel algorithm. We next identify certain switches and pairs of switches that allow one to move from any Hankel tournament $T \in \mathcal{T}_H(R)$ to T^* where each switch and pairs of switches produces another Hankel tournament in $\mathcal{T}_H(R)$. Since these switches are reversible, this allows one to move from any $T_1 \in \mathcal{T}_H(R)$ to any other $T_2 \in \mathcal{T}_H(R)$ where each switch produces a Hankel tournament in $\mathcal{T}_H(R)$. We collect all these switches, including the switches used for loopy tournament and a switch to be used for combinatorially skew-Hankel tournaments, in Table 1.

Let i, j, k, l be four distinct indices and consider the 4×4 submatrix $T[i, j, k, l]$ of T whose row and column indices are $\{i, j, k, l\}$. If we have a directed cycle $i \to j \to k \to l \to i$ in a tournament T, then $T[i, j, k, l]$ has the form on the left in (10), and we define $\square_{i,j,k,l}$ to be the switch that replaces $T[i, j, k, l]$ with the matrix on the right:

		i	j	k	l				i	j	k	l
i			1	a	0		i			0	a	1
j		0		1	b	$\to$	j		1		0	b
k		$1-a$	0		1		k		$1-a$	1		0
l		1	$1-b$	0			l		0	$1-b$	1	

$$(10)$$

The switch $\square_{i,j,k,l}$ reverses a 4-cycle of T:

$$(i \to j \to k \to l \to i) \to (i \leftarrow j \leftarrow k \leftarrow l \leftarrow i)$$

and we call it a *4-cycle switch*. It is easy to see that this reversal of a 4-cycle can also be accomplished by the reversal of two 3-cycles: if, for instance, we have $a = b = 0$, we reverse two 3-cycles using the *3-cycle switches* $\triangle_{i,j,k}$ and then $\triangle_{i,k,l}$. We define $\triangle^c_{i,j,k}$ to be the switch $\triangle_{n+1-i,n+1-j,n+1-k}$ and we refer to it as the *complementary switch* of $\triangle_{i,j,k}$ (the indices are complementary to i, j, k). In the same way, we define $\square^c_{i,j,k,l}$ to be the switch $\square_{n+1-i,n+1-j,n+1-k,n+1-l}$ and call it the *complementary switch* of $\square_{i,j,k,l}$.

For Hankel tournaments, we have that

$$T[i, j, k, l] = T[n + 1 - l, n + 1 - k, n + 1 - j, n + 1 - i]^h.$$

We will consider two possibilities for the indices i, j, k, l:

(i) $j \neq n + 1 - i, n + 1 - k$ and $l \neq n + 1 - i, n + 1 - k$ with at most one of the equalities $k = n + 1 - i$ and $l = n + 1 - j$;

(ii) $k = n + 1 - j$ and $l = n + 1 - i$.

Table 1
Summary of switches.

edge-loop switch	$\rightarrowtail_{ij}$	$\begin{array}{c\|c\|c} & i & j \\ \hline i & 0 & 1 \\ \hline j & 0 & 1 \end{array} \rightarrow \begin{array}{c\|c\|c} & i & j \\ \hline i & 1 & 0 \\ \hline j & 1 & 0 \end{array}$	
3-cycle switch	$\triangle_{i,j,k}$	$\begin{array}{c\|c\|c\|c} & i & j & k \\ \hline i & x & 1 & 0 \\ \hline j & 0 & y & 1 \\ \hline k & 1 & 0 & z \end{array} \rightarrow \begin{array}{c\|c\|c\|c} & i & j & k \\ \hline i & x & 0 & 1 \\ \hline j & 1 & y & 0 \\ \hline k & 0 & 1 & z \end{array}$	
3-cycle switch	$\triangle_{i,j,k}$	no loops: $x = y = z = 0$ above	
3-cycle Hankel switch	$\triangle_{i,\frac{n+1}{2},n+1-i}$	$j = (n+1)/2$ and $k = n+1-i$ above	
4-cycle switch	$\square_{i,j,k,l}$	$\begin{array}{c\|c\|c\|c\|c} & i & j & k & l \\ \hline i & 0 & 1 & a & 0 \\ \hline j & 0 & 0 & 1 & b \\ \hline k & 1-a & 0 & 0 & 1 \\ \hline l & 1 & 1-b & 0 & 0 \end{array} \rightarrow \begin{array}{c\|c\|c\|c\|c} & i & j & k & l \\ \hline i & 0 & 0 & a & 1 \\ \hline j & 1 & 0 & 0 & b \\ \hline k & 1-a & 1 & 0 & 0 \\ \hline l & 0 & 1-b & 1 & 0 \end{array}$	
4-cycle Hankel switch	$\square_{i,j,n+1-j,n+1-i}$	$k = n+1-j$, $l = n+1-i$, and $a = b$ above	
4-cycle skew-Hankel switch	$\square_{i,j,n+1-i,n+1-j}$	$\begin{array}{c\|c\|c\|c\|c} & i & j & k & l \\ \hline i & 0 & 1 & 0 & 0 \\ \hline j & 0 & 0 & 1 & 0 \\ \hline k & 0 & 0 & 0 & 1 \\ \hline l & 1 & 0 & 0 & 0 \end{array} \rightarrow \begin{array}{c\|c\|c\|c\|c} & i & j & k & l \\ \hline i & 0 & 0 & 0 & 1 \\ \hline j & 1 & 0 & 0 & 0 \\ \hline k & 0 & 1 & 0 & 0 \\ \hline l & 0 & 0 & 1 & 0 \end{array}$	$k = n+1-i$ $l = n+1-j$

In case (i), for n even, there is no overlap on the entries of $T[i, j, k, l]$ and $T[n + 1 - l, n + 1 - k, n + 1 - j, n + 1 - i]$ in the positions we are changing when applying $\square_{i,j,k,l}$ and $\square^c_{l,k,j,i}$. We say that $\square_{i,j,k,l}$ is a *pure 4-cycle switch*. Applying the pure 4-cycle switch $\square_{i,j,k,l}$ and the complementary switch in the reverse order $\square^c_{l,k,j,i} = \square_{n+1-l,n+1-k,n+1-j,n+1-i}$ to a $T \in \mathcal{T}_H(R)$ results in another Hankel tournament in $\mathcal{T}_H(R)$.

In case (ii), the matrix $T[i, j, n + 1 - j, n + 1 - i]$ has the form

	i	j	$n+1-j$	$n+1-i$
i	0	1	a	0
j	0	0	1	a
$n+1-j$	$1-a$	0	0	1
$n+1-i$	1	$1-a$	0	0

.

If we apply $\square_{i,j,n+1-j,n+1-i}$ to a $T \in \mathcal{T}_H(R)$ we obtain another Hankel tournament in $\mathcal{T}_H(R)$. In this case $\square_{i,j,n+1-j,n+1-i} = \square^c_{n+1-i,n+1-j,j,i}$ and we just need to apply one 4-cycle switch.

If n is odd and $j = \frac{n+1}{2}$, then $T[i, j, n + 1 - j, n + 1 - i]$ collapses to the 3×3 matrix $T[i, \frac{n+1}{2}, n + 1 - i]$, and if there is a 3-cycle we can apply the 3-cycle switch $\triangle_{i,\frac{n+1}{2},n+1-i}$, which reverses the orientation of the 3-cycle. We call such a switch a 3-*cycle Hankel switch*, when we have these special indices. Applying a 3-cycle Hankel switch $\triangle_{i,\frac{n+1}{2},n+1-i}$ to a $T \in \mathcal{T}_H(R)$ results in another Hankel tournament in $\mathcal{T}_H(R)$.

When R is nondecreasing, we now show how to use these switches to move from any tournament $T \in \mathcal{T}_H(R)$ to $T^* \in \mathcal{T}_H(R)$ never leaving the class $\mathcal{T}_H(R)$. We consider two cases according to whether n is even or odd.

Case n even: If the first column of T equals the first column of T^*, then their last columns, first rows, and last rows are equal, respectively. Thus the *borders* of T and T^* agree; deleting their borders, we proceed by induction on n. Now suppose that the borders of T and T^* are not equal. The first column of T^* consists of $n - s_1 = r_1 + 1$ 0s followed by s_1 1s. Since $t_{11} = 0$, there is a first $i \neq 1$ such that $t_{i1} = 1$ and $t_{j1} = 0$ for some $j > i$.

First suppose that we may take $j = n$, that is, $t_{n1} = 0$. Since r_1 is the smallest score, and $r_1 + s_1 = n - 1$, we have that $s_1 \geq (n/2)$. Since $t_{n1} = 0$, it follows that there exists an $1 < i \leq (n/2)$ such that $t_{i1} = t_{n+1-i,1} = 1$. Since T is a Hankel

tournament, we have

$$T[1, i, n+1-i, n] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 1 & 1-a & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

that is,

$$(a=0) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{or} \quad (a=1) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

In the case that $a = 0$, we apply the 4-cycle Hankel switch $\square_{1,n,n+1-i,i}$ and the result is a tournament in $\mathcal{T}_H(R)$ in which t_{i1} has been replaced with 0 and t_{n1} has been replaced with 1. In the case that $a = 1$, we apply the 4-cycle Hankel switch $\square_{1,n,i,n+1-i}$ and the result is a tournament in $\mathcal{T}_H(R)$ in which $t_{n+1-i,1}$ has been replaced with 0 and t_{n1} has been replaced with 1.

Now assume that $t_{n1} = 1$ so that there exists a j with $i < j < n$ such that $t_{j1} = 0$. First suppose that we may take $j = n + 1 - i$. Since $r_i < r_{n+1-i}$ there exists a k with $k \neq 1, n + 1 - i$, such that $t_{ik} = 0$ and $t_{n+1-i,k} = 1$. Since $t_{i1} = 1$ and T is a Hankel tournament, $t_{n,n+1-i} = 1$ and hence $t_{n+1-i,n} = 0$. Therefore $k \neq n$. In addition, because $r_i < r_{n+1-i}$, we may choose $k \neq i$. We now have

$$T[1, i, n+1-i, k] = \begin{bmatrix} 0 & 0 & 1 & a \\ 1 & 0 & b & 0 \\ 0 & 1-b & 0 & 1 \\ 1-a & 1 & 0 & 0 \end{bmatrix}.$$

The switch $\square_{1,n+1-i,k,i}$ is a pure 4-cycle switch, and $\square_{1,n+1-i,k,i}$ followed by $\square^c_{i,k,n+1-i,1}$ produces a tournament in $\mathcal{T}_H(R)$ in which t_{i1} has been replaced with 0 and $t_{n+1-i,1}$ has been replaced with 1.

The remaining possibility in the n even case is that $t_{i1} = 1$ implies that $t_{n+1-i,1} = 1$, so we cannot take $j = n + 1 - i$. If $t_{ji} = 1$, then

$$T[1, i, j] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The switch $\triangle_{1,j,i}$ is a pure 3-cycle switch, and $\triangle_{1,j,i}$ followed by $\triangle^c_{i,j,1}$ produces a tournament in $\mathcal{T}_H(R)$ in which t_{i1} has been replaced with 0 and t_{j1} has been replaced with 1. Now suppose that $t_{ji} = 0$ so that $t_{ij} = 1$. Since $r_i \leq r_j$, there exists at least two values of k such that $k \neq 1, i, j$, and $t_{ik} = 0$ and $t_{jk} = 1$. If $t_{j,n+1-i} = 0$, then there exists such a $k \neq n + 1 - i, n + 1 - j$. If $t_{j,n+1-i} = 1$, then since T is a Hankel tournament, $t_{i,n+1-j} = 1$ and hence $k \neq n + 1 - j$, and since there are two possible values of k, we may choose $k \neq n + 1 - i$. Thus we have

$$T[1, i, j, k] = \begin{bmatrix} 0 & 0 & 1 & a \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1-a & 1 & 0 & 0 \end{bmatrix}.$$

The switch $\square_{1,j,k,i}$ is a pure 4-cycle switch, and $\square_{1,j,k,i}$ followed by $\square^c_{i,k,j,1}$ produces a tournament in $\mathcal{T}_H(R)$ in which t_{i1} has been replaced with 0 and t_{j1} has been replaced with 1.

Hence by induction, if n is even, we can move from T to T^* by a sequence of 4-cycle Hankel switches, pairs of switches consisting of a pure 4-cycle switch $\square_{i,j,k,l}$ and its complementary switch in the reverse order $\square^c_{l,k,j,i}$, and pairs of switches consisting of a pure 3-cycle switch and its complementary switch in the reverse order.

Case n odd: Then row $(n+1)/2$ of the matrix T^* constructed by the Hankel algorithm contains $(n-1)/2$ 1s and they occur in its first $(n-1)/2$ columns, thus determining both row and column $(n+1)/2$. We show that by 3-cycle Hankel switches and pairs of switches consisting of a pure 4-cycle switch and the complementary switch in the reverse order, we can bring any tournament $T \in \mathcal{T}_H(R)$ into a tournament $T' \in \mathcal{T}_H(R)$ which agrees with T^* on row $(n+1)/2$, that is, has all its 1s in columns $1, 2, \ldots, (n-1)/2$. We may then delete row and column $(n+1)/2$ of T' and T^* leaving two $(n-1) \times (n-1)$ Hankel tournaments T'' and T^{**} with the same score vector R'. By what we have proved, T'' can be brought to T^{**} by a sequence of 4-cycle Hankel switches, pairs of switches consisting of a pure 4-cycle switch and its complementary switch in the reverse order, and pairs of switches consisting of a pure 3-cycle switch and its complementary switch in the reverse order. It then follows that T can be brought to T^* by a sequence of 3-cycle Hankel switches, 4-cycle Hankel switches, and pairs of switches consisting of a pure 4-cycle switch and its complementary switch in the reverse order.
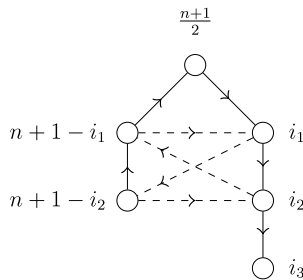
Fig. 1. Part of the digraph D.

So suppose that $t_{(n+1)/2,i} = 0$ for some $i \leq (n-1)/2$. We consider the digraph D with vertex set $\{1, 2, \ldots, n\}$ with an edge from i to j if and only if $t_{ij} = 1$ and $t_{ij}^* = 0$. Since T and T^* have the same score vector, the outdegree of a vertex equals its indegree and hence the edges of D can be partitioned into cycles. One of these cycles C uses the vertex $(n+1)/2$:

$$\frac{n+1}{2} \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k \rightarrow \frac{n+1}{2}.$$

Since we are dealing with Hankel tournaments, D also contains the edge $(n+1-i_1) \rightarrow \frac{n+1}{2}$. If T also contains the edge $i_1 \rightarrow (n+1-i_1)$, then we can apply the 3-cycle Hankel switch $\triangle_{n+1-i_1, \frac{n+1}{2}, i_1}$ and move T closer to T^*. If T contains the edge $(n+1-i_1) \rightarrow i_1$ (and so does not contain the edge $i_1 \rightarrow (n+1-i_1)$), then we consider the edge $i_1 \rightarrow i_2$. Then $n+1-i_2 \rightarrow n+1-i_1$ is also an edge of D. If $i_2 \rightarrow (n+1-i_2)$ in T, then the 4-cycle Hankel switch $\square_{n+1-i_1, i_1, i_2, n+1-i_2}$ reverses $(n+1-i_1) \rightarrow i_1$, and then we can apply $\triangle_{n+1-i_1, \frac{n+1}{2}, i_1}$ as above. If, on the other hand, $(n+1-i_2) \rightarrow i_2$ is an edge of T, we consider the pair of vertices $(n+1-i_1)$ and i_2 (see Fig. 1), and proceed as follows.

If $i_2 \rightarrow (n+1-i_1)$ is an edge of T, then we apply the 4-cycle Hankel switch $\square_{i_2, n+1-i_1, i_1, n+1-i_2}$ to reverse $(n+1-i_1) \rightarrow i_1$ and then $\triangle_{n+1-i_1, \frac{n+1}{2}, i_1}$ as above.

If $(n+1-i_1) \rightarrow i_2$ is an edge of T, then we consider the edge $i_2 \rightarrow i_3$ where now $i_3 \neq (n+1-i_1), (n+1-i_2)$.

We continue like this until we are able to find an $i_j \neq n+1-i_1, \ldots, n+1-i_{j-1}$ such that $i_j \rightarrow (n+1-i_h)$ for some $h \leq j$. There are two possibilities to consider. If we also have the edge $(n+1-i_j) \rightarrow i_j$ (only if $h < j$), then we apply the 4-cycle Hankel switch $\square_{i_h, n+1-i_j, i_j, n+1-i_h}$ to reverse $(n+1-i_h) \rightarrow i_h$, followed by the 4-cycle Hankel switches $\square_{i_{l-1}, i_l, n+1-i_l, n+1-i_{l-1}}$ with $l = h, h-1, \ldots, 2$ to finally reverse $(n+1-i_1) \rightarrow i_1$, and then $\triangle_{n+1-i_1, \frac{n+1}{2}, i_1}$. If for all j with $2 \leq j \leq k$, $i_j \rightarrow (n+1-i_h)$ is not an edge of T for a h with $1 \leq h \leq j$, then since $i_k \rightarrow (n+1)/2$ is an edge of cycle C, the cycle C does not contain both a vertex p and a vertex $n+1-p$. Thus using a sequence of pairs of pure 3-cycle switches and their complementary 3-cycle switches in the reverse order, we can reverse C and its complementary cycle of opposite orientation. By these operations we have obtained from T a Hankel tournament in $\mathcal{T}_H(R)$ which agrees with T^* in row and column $(n+1)/2$.

Putting the preceding arguments together and having in mind that a pure 4-cycle switch can be accomplished by two 3-cycle switches, we have proved the following theorem.

Theorem 15. *Let T_1 and T_2 be two tournaments in $\mathcal{T}_H(R)$. Then there exists a sequence of moves consisting of 4-cycle Hankel switches, pairs consisting of a pure 3-cycle switch and its complementary 3-cycle switch in the reverse order, and 3-cycle Hankel switches, which brings T_1 to T_2 with all intermediary tournaments in $\mathcal{T}_H(R)$.*

Example 16. Let $n = 7$ and $R = (1, 2, 2, 3, 4, 4, 5)$, and consider the two Hankel tournaments T_1 and T_2 in $\mathcal{T}_H(R)$ given by

$$
T_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
T_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
$$

We can move from T_1 to T_2 by the 4-cycle Hankel switch $\square_{7,5,3,1}$, followed by the pair of pure 3-cycle switches $\triangle_{3,2,1}$ and $\triangle_{1,2,3}^c = \triangle_{7,6,5}$, followed by the 3-cycle Hankel switch $\triangle_{6,4,2}$.

For completeness we mention the following. It is natural as well to consider $n \times n$ Hankel loopy tournaments $T = [t_{ij}]$, that is, Hankel tournaments with possible 1s on the main diagonal. By the Hankel property, we now have that for all i, $t_{ii} = t_{n+1-i,n+1-i}$ where the common value is either 0 or 1. The score vector $R = (r_1, r_2, \ldots, r_n)$ of a Hankel loopy tournament satisfies $r_i + r_{n+1-i} = n+1$ or $n-1$ depending on whether or not $t_{ii} = t_{n+1-i,n+1-i} = 1$. Thus the score vector of a Hankel loopy tournament determines which elements on the main diagonal equal 1 and which equal 0. Thus if R' is obtained from

R by subtracting 1 from those i and $n + 1 - i$ for which $r_i + r_{n+1-i} = n + 1$, then there is a Hankel loopy tournament with score vector R if and only if there is a Hankel tournament with score vector R'.

Example 17. The Hankel loopy tournament

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

with score vector $R = (2, 3, 3, 3, 2)$ corresponds to the Hankel tournament

$$T' = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

with score vector $R' = (2, 2, 2, 2, 2)$.

4. Combinatorially skew-Hankel tournaments

Let $T = [t_{ij}]$ be an $n \times n$ combinatorially skew-Hankel tournament, and let the score vector of T be $R = (r_1, r_2, \ldots, r_n)$. Since T is invariant under a rotation by $180°$, we have that for each i, row $n + 1 - i$ is obtained by reversing row i, and thus the score vector of T is *palindromic*, that is,

$$R = (r_1, r_2, r_3, \ldots, r_3, r_2, r_1).$$

Since T is combinatorially skew with respect to the main diagonal, row i of T not only determines row $n + 1 - i$ but it also determines columns i and $n + 1 - i$. We illustrate this important property in the next example.

Example 18. Let T be an 8×8 combinatorially skew-Hankel tournament with row 3 given. Then the entries in column 3, and row and column $8 + 1 - 3 = 6$ are as shown in

$$\begin{bmatrix} 0 & & 1-a & & & 1-f & & 0 \\ & 0 & 1-b & & & 1-e & 0 & \\ a & b & 0 & c & d & 0 & e & f \\ & & 1-c & 0 & 0 & 1-d & & \\ & & 1-d & 0 & 0 & 1-c & & \\ f & e & 0 & d & c & 0 & b & a \\ & 0 & 1-e & & & 1-b & 0 & \\ 0 & & 1-f & & & 1-a & & 0 \end{bmatrix},$$

where the 0s on the main and Hankel diagonals have been inserted.

The score vector of a combinatorially skew-Hankel tournament is determined by $n/2$ integers if n is even, and by $(n+1)/2$ integers if n is odd. For n even, if we reorder the rows $1, 2, \ldots, n/2$ and then the rows $n/2 + 1, n/2 + 2, \ldots, n$ in the same way, and similarly reorder columns $1, 2, \ldots, n/2$ and columns $n/2 + 1, n/2 + 2, \ldots, n$ in the corresponding way, the result is another combinatorially skew-Hankel tournament. For n odd, if we do a reordering for the indices $1, 2, \ldots, (n - 1)/2$ and $(n + 1)/2, (n + 1)/2 + 1, \ldots, n$, the result is also another combinatorially skew-Hankel tournament. Hence there is no loss of generality in assuming that the first half $(r_1, r_2, \ldots, r_{\lfloor n/2 \rfloor})$ of R is nondecreasing and thus that the second half $(r_{\lceil (n+2)/2 \rceil}, \ldots, r_2, r_1)$ is nonincreasing. If n is odd, R has a middle term $r_{(n+1)/2}$ which is even since $t_{(n+1)/2,(n+1)/2} = 0$, and both row $(n + 1)/2$ and column $(n + 1)/2$ of T are palindromic.

Example 19. A 7×7 combinatorially skew-Hankel tournament with palindromic score vector $R = (1, 3, 4, 2, 4, 3, 1)$ is given by

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We first treat the case where n is even. In this case a combinatorially skew-Hankel tournament is of the form

$$\begin{bmatrix} T_1 & T_2 \\ T_2^{th} & T_1^{ht} \end{bmatrix}$$

where T_1 is an $\frac{n}{2} \times \frac{n}{2}$ tournament and T_2, when its columns are taken from last to first, is an $\frac{n}{2} \times \frac{n}{2}$ tournament.

Let $R = (r_1, r_2, \ldots, r_{n/2}, r_{n/2}, \ldots, r_2, r_1)$ satisfy $r_1 \leq r_2 \leq \cdots \leq r_{n/2}$. The nondecreasing rearrangement of R is $R^* = (r_1^*, r_2^*, \ldots, r_n^*) = (r_1, r_1, r_2, r_2, \ldots, r_{n/2}, r_{n/2})$ where therefore $r_{2k-1}^* = r_{2k}^* = r_k$ for $k = 1, 2, \ldots, n/2$. Since in an $n \times n$ combinatorially skew-Hankel tournament $T = [t_{ij}]$, $t_{kk} = t_{k,n+1-k} = t_{n+1-k,n+1-k} = t_{n+1-k,k} = 0$, the sum of the l smallest elements of R, that is, the first l elements of R^*, must be at least the number $\binom{l}{2} - \lfloor \frac{l}{2} \rfloor$ of games played amongst themselves, and thus

$$\sum_{i=1}^{l} r_i^* \geq \binom{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor, \quad (l = 1, 2, \ldots, n), \text{ with equality if } l = n. \tag{11}$$

Lemma 20. *Let h, with $1 \leq h \leq n/2$, be such that equality holds in (11) for $l = 2h - 1$. Then equality also holds in (11) for $l = 2h$ and $l = 2h - 2$.*

Proof. We have $\sum_{i=1}^{2h-1} r_i^* = \binom{2h-1}{2} - (h-1)$. If $h = 1$, then $r_1^* = 0$ and hence $r_2^* = 0$, and so equality holds for $l = 2$. Now let $h \geq 2$. Then

$$r_{2h}^* = \sum_{i=1}^{2h} r_i^* - \sum_{i=1}^{2h-1} r_i^* \geq \binom{2h}{2} - h - \left(\binom{2h-1}{2} - (h-1) \right) = 2h - 2,$$

and

$$r_{2h-1}^* = \sum_{i=1}^{2h-1} r_i^* - \sum_{i=1}^{2h-2} r_i^* \leq \left(\binom{2h-1}{2} - (h-1) \right) - \left(\binom{2h-2}{2} - (h-1) \right) = 2h - 2. \tag{12}$$

Thus $2h - 2 \leq r_{2h}^* = r_h = r_{2h-1}^* \leq 2h - 2$, and we conclude that $r_{2h}^* = 2h - 2$. With this value for r_{2h}^*, we get

$$\sum_{i=1}^{2h} r_i^* = \left(\sum_{i=1}^{2h-1} r_i^* \right) + r_{2h}^* = \binom{2h-1}{2} - (h-1) + (2h-2) = \binom{2h}{2} - h,$$

and thus equality holds in (11). Because we have obtained $r_{2h-1}^* = r_{2h}^* = 2h - 2$, we have equality in (12), which yields

$$\sum_{i=1}^{2h-2} r_i^* = \binom{2h-2}{2} - (h-1)$$

as was claimed. $\square$

By taking $l = 2k$ in (11), we see that the inequalities (11) for l even are equivalent to

$$\sum_{i=1}^{k} r_i \geq k(k-1), \quad (k = 1, 2, \ldots, \frac{n}{2}), \text{ with equality for } k = \frac{n}{2}. \tag{13}$$

In the next lemma we show that (11) is equivalent to (13) even if l is odd.

Lemma 21. *The vector R^* satisfies (11) if and only if R satisfies (13).*

Proof. The inequalities (11) for l even are equivalent to the inequalities (13) for $k = 1, 2, \ldots, n/2$. Thus we need only show that if the inequalities (13) hold, then the inequalities (11) hold for odd l. Let $l = 2k + 1$ and suppose to the contrary that $\sum_{i=1}^{2k+1} r_i^* < \binom{2k+1}{2} - k$. Then

$$r_{2k+1}^* = \sum_{i=1}^{2k+1} r_i^* - \sum_{i=1}^{2k} r_i^* < \left(\binom{2k+1}{2} - k \right) - \left(\binom{2k}{2} - k \right) = 2k.$$

Since $r_{2k+2}^* = r_{2k+1}^*$, we obtain

$$\sum_{i=1}^{2k+2} r_i^* = \left(\sum_{i=1}^{2k+1} r_i^* \right) + r_{2k+2}^* < \left(\binom{2k+1}{2} - k \right) + 2k = \binom{2k+2}{2} - (k+1),$$

and this contradicts (11) and (13). $\square$

We now verify that it is enough for us to assume that $(r_1, r_2, \ldots, r_{(n+1)/2})$ is 3-nearly nondecreasing.

Lemma 22. *Let $R = (r_1, r_2, \ldots, r_m)$ be a sequence of nonnegative integers that satisfies* (13) *such that R is 3-nearly nondecreasing. Then its nondecreasing rearrangement also satisfies* (13).

Proof. It is enough to show that if for some k, $r_k = a$ and $r_{k+1} = a - p$ where $1 \le p \le 3$, then interchanging r_k and r_{k+1} results in a vector also satisfying (13). If this was false, then we have that $\sum_{i=1}^{k} r_i = k(k-1) + h$ where $0 \le h \le p - 1$. Then

$$r_{k+1} = \sum_{i=1}^{k+1} r_i - \sum_{i=1}^{k} r_i \ge (k+1)k - k(k-1) - h = 2k - h.$$

But we also have

$$r_k = \sum_{i=1}^{k} r_i - \sum_{i=1}^{k-1} r_i \le k(k-1) + h - (k-1)(k-2) = 2k - 2 + h.$$

Therefore

$$(2k - h) + p \le r_{k+1} + p = r_k \le 2k - 2 + h,$$

implying that $h \ge (p/2) + 1$, a contradiction since $p \le 3$. □

Theorem 23. *Let n be an even integer, and let $R = (r_1, r_2, \ldots, r_{n/2}, r_{n/2}, \ldots, r_2, r_1)$ be a vector of nonnegative integers such that $(r_1, r_2, \ldots, r_{n/2})$ is nondecreasing. Then there exists a combinatorially skew-Hankel tournament with score vector R if and only if* (13) *holds.*

Proof. We have already verified that (13) holds for a combinatorially skew-Hankel tournament with score vector R. For the converse, assume that (13) holds but there does not exist a combinatorially skew-Hankel tournament with score vector R. We let R be a counterexample with n minimum and for this n with r_1 minimum.

As above, we let R^* be the nondecreasing vector $(r_1, r_1, r_2, r_2, \ldots, r_{n/2}, r_{n/2})$. By Lemma 21, R^* satisfies (11). By Lemma 20 if equality holds in (11) for an odd integer, it also holds for the next even integer. We consider two cases depending on whether or not equality holds in (11) for some even integer $l = 2h < n$.

Case 1: There exists an integer $l = 2h < n$ such that equality holds in (11).

Let $R' = (r_1, \ldots, r_h, r_h, \ldots, r_1)$. Since $r_{h+1} \ge 2h$, the vector

$$R'' = (r_1'', \ldots, r_{(n/2)-h}'', r_{(n/2)-h}'', \ldots, r_1'') = (r_{h+1} - 2h, \ldots, r_{n/2} - 2h, r_{n/2} - 2h, \ldots, r_{h+1} - 2h)$$

is nonnegative. Then R' satisfies the inequality in (13) for $1 \le k \le h$ with equality if $k = h$. Also

$$\sum_{i=1}^{k} (r_i'' + 2h) = \sum_{i=1}^{h+k} r_i - \sum_{i=1}^{h} r_i \ge (h+k)(h+k-1) - h(h-1) = k(k-1) + 2hk,$$

with equality when $k = (n/2) - h$. Thus

$$\sum_{i=1}^{k} r_i'' \ge k(k-1) \quad \text{with equality if } k = (n/2) - h,$$

and R'' satisfies the corresponding condition (13). By our minimality assumption on n, there exist combinatorially skew-Hankel tournaments T' and T'' with score vectors R' and R'', respectively. We can write

$$T' = \left[\begin{array}{c|c} T_1' & T_2' \\ \hline (T_2')^{th} & (T_1')^{th} \end{array} \right],$$

where T_1' and T_2' are $h \times h$. Then

$$T = \left[\begin{array}{c|c|c} T_1' & O_{h,n-2h} & T_2' \\ \hline J_{n-2h,h} & T'' & J_{n-2h,h} \\ \hline (T_2')^{th} & O_{h,n-2h} & (T_1')^{th} \end{array} \right]$$

is a combinatorially skew-Hankel tournament with score vector R, a contradiction.

Case 2: There does not exists an integer $l < n$ such that equality holds in (11).

We first claim that the difference between the left hand side of (11) and its right hand side is at least 2 for $l < n$. If l is even both sides are even so this holds. Now suppose that $l = 2k + 1$ and, to the contrary, that

$$\sum_{i=1}^{2k+1} r_i^* = \left(\binom{2k+1}{2} - k \right) + 1.$$

We calculate that

$$r^*_{2k+1} = \sum_{i=1}^{2k+1} r^*_i - \sum_{i=1}^{2k} r^*_i \le \left(\binom{2k+1}{2} - k \right) + 1 - \left(\binom{2k}{2} - k + 2 \right) = 2k - 1,$$

and

$$r^*_{2k+2} = \sum_{i=1}^{2k+2} r^*_i - \sum_{i=1}^{2k+1} r^*_i \ge \left(\binom{2k+2}{2} - (k+1) + 2 \right) - \left(\binom{2k+1}{2} - k + 1 \right) = 2k + 1.$$

Thus

$$2k + 1 \le r^*_{2k+2} = r_{k+1} = r^*_{2k+1} \le 2k - 1,$$

a contradiction. This verifies our claim. Now we consider the palindromic $R' = (r'_1, \ldots, r'_{n/2}, r'_{n/2}, \ldots, r'_1)$ where $r'_1 = r_1 - 1$, $r'_{n/2} = r_{n/2} + 1$, and $r'_i = r_i$ for $2 \le i \le (n/2) - 1$. Then the first half of R' is nondecreasing. Since equality does not occur in (11) for $l < n$, we have $r_1 \ge 1$, and so $r'_1 \ge 0$, and for $l \ge 2$, the conditions corresponding to (11) hold for R'. By the minimality assumption on r_1, there is a combinatorially skew-Hankel tournament $T' = [t'_{ij}]$ with score vector R'. Since $r'_{n/2} - r'_1 \ge 2$, there is a column $p \ne 1$ such that $t'_{1p} = 0$ and $t'_{n/2,p} = 1$. Since T' is a combinatorially skew-Hankel tournament, we have $p \ne n/2, (n/2) + 1$. If $p \ne n$, then without loss of generality we may assume that $p < n/2$. Then the 6×6 principal submatrix $T'[1, p, n/2, n/2 + 1, n + 1 - p, n]$ has the form

	1	p	$n/2$	$n/2+1$	$n+1-p$	n
1	0	0				0
p	1	0	0		0	
$n/2$		1	0	0		
$n/2+1$			0	0	1	
$n+1-p$		0		0	0	1
n	0			0	0	0

If we now change 0s to 1s and 1s to 0s in the shaded cells in the positions of T', we obtain a combinatorially skew-Hankel tournament with score vector R, a contradiction.

If there is no such p strictly between 1 and n, and the only $p \ne 1$ with $t'_{1p} = 0$ and $t_{n/2,p} = 1$ is $p = n$, then we have the 4×4 principal submatrix

$$T[1, n/2, n/2 + 1, n] = \begin{bmatrix} 0 & a & 0 & 0 \\ a' & 0 & 0 & 1 \\ 1 & 0 & 0 & a' \\ 0 & 0 & a & 0 \end{bmatrix}$$

where $a' = 1 - a$. If $a = 1$, then $r'_1 \ge r'_{n/2}$ which is impossible. Thus $a = 0$ and $a' = 1$. Interchanging the 0's and 1's given by a and a' gives a combinatorially skew-Hankel tournament with score vector R, a contradiction again. $\square$

In case n is odd we have the following characterization of score vectors of $n \times n$ combinatorially skew-Hankel tournaments.

Theorem 24. *Let n be an odd integer, and let $R = (r_1, \ldots, r_{(n-1)/2}, r_{(n+1)/2}, r_{(n-1)/2}, \ldots, r_1)$ be a vector of nonnegative integers such that $(r_1, r_2, \ldots, r_{(n-1)/2})$ is nondecreasing. Then there exists a combinatorially skew-Hankel tournament with score vector R if and only if*

$$r_{(n+1)/2} \le n - 1 \tag{14}$$

and

$$\sum_{i=1}^{k} r_i \ge k(k-1) + \left(k - \frac{r_{(n+1)/2}}{2} \right)^+, \quad \left(k = 1, 2, \ldots, \frac{n-1}{2} \right), \quad \text{with equality if } k = \frac{n-1}{2}. \tag{15}$$

Proof. The inequality (14) is certainly necessary for the existence of a $T \in \mathcal{T}_{H^*}(R)$. The inequalities (15) are also necessary because in the $2k$ rows of T with the smallest row sums, the minimum number of 1s that could be in column $(n+1)/2$ is $(2k - r_{(n+1)/2})^+$.

Now assume that (14) and (15) hold. If we take an $n \times n$ combinatorially skew-Hankel tournament T with score vector R and delete the middle row and middle column, we are left with an $(n-1) \times (n-1)$ combinatorially skew-Hankel tournament with a nearly nondecreasing score vector. The conditions (15) imply that if we choose the middle column to have its 1s in the rows with the largest row sums (thereby determining the middle column) and delete the middle row and middle column,

the row sum vector of the resulting matrix is a nearly nondecreasing vector R' satisfying the conditions corresponding to (13); by Lemma 22, the nondecreasing rearrangement R'' of R' also satisfies the same conditions. Hence by Theorem 23 there exists a combinatorially skew-Hankel tournament with score vector R'' and thus one with score vector R'. Hence there is a combinatorially skew-Hankel tournament with score vector R. $\square$

We now describe an algorithm to produce a combinatorially skew-Hankel tournament with a prescribed score vector R when the conditions of Theorems 23 and 24 hold.

Skew-Hankel Algorithm for a $T = [t_{ij}] \in \mathcal{T}_{H^*}(R)$

Let $R_n = (r_1, r_2, \ldots, r_n)$ be a vector of nonnegative integers such that $r_i = r_{n+1-i}$ for all i and $r_1 \leq r_2 \leq \cdots \leq r_{\lfloor \frac{n}{2} \rfloor}$. If n is even, assume that (13) holds. If n is odd, assume that (15) holds.

1. If n is odd:
   (a) Let $v = (v_1, v_2, \ldots, v_n)$ be a (0, 1)-vector with $v_i = v_{n+1-i} = 1$ if $\frac{r_{(n+1)/2}}{2} + 1 \leq i \leq \frac{n-1}{2}$ and $v_i = v_{n+1-i} = 0$ otherwise, and let the nearly nondecreasing vector $R'_{n-1} = (r'_1, r'_2, \ldots, r'_{n-1})$ be defined by $r'_i = r'_{(n-1)+1-i} = r_i - v_i$ for $1 \leq i \leq (n-1)/2$.
   (b) Let Q_{n-1} be an $(n-1) \times (n-1)$ permutation matrix of the form

   $$\begin{bmatrix} P & O_{(n-1)/2,(n-1)/2} \\ O_{(n-1)/2,(n-1)/2} & P^{th} \end{bmatrix}$$

   such that the first half of $R_{n-1} = R'_{n-1} Q^t_{n-1}$ is nondecreasing.
   (c) Let T_{n-1} be a combinatorially skew-Hankel tournament with score vector R_{n-1} obtained by applying this algorithm, and let T'_{n-1} be the combinatorially skew-Hankel tournament with score vector R'_{n-1} written in the form

   $$T'_{n-1} = Q^t_{n-1} T_{n-1} Q_{n-1} = \left[\begin{array}{c|c} A & B \\ \hline B^{th} & A^{th} \end{array} \right].$$

   (d) Let the combinatorially skew-Hankel matrix T_n be defined by

   $$T_n = \left[\begin{array}{ccc|c|ccc}
   & & & v_1 & & & \\
   & A & & \vdots & & B & \\
   & & & v_{(n-1)/2} & & & \\
   \hline
   (1-v_1) & \cdots & (1-v_{(n-1)/2}) & 0 & (1-v_{(n+3)/2}) & \cdots & (1-v_n) \\
   \hline
   & & & v_{(n+3)/2} & & & \\
   & B^{th} & & \vdots & & A^{th} & \\
   & & & v_n & & & \\
   \end{array} \right].$$

2. If n is even:
   (a) Let $v = (v_1, v_2, \ldots, v_n)$ be a (0, 1)-vector with $v_i = 1$ if and only if $2 + \lceil \frac{r_1}{2} \rceil \leq i \leq n - 1 - \lfloor \frac{r_1}{2} \rfloor$, and let the vector $R'_{n-2} = (r'_1, r'_2, \ldots, r'_{n-2})$, the first half of which is 2-nearly nondecreasing, be defined by $r'_i = r'_{(n-2)+1-i} = r_{i+1} - v_{i+1} - v_{n-i}$ for $1 \leq i \leq n-2$.
   (b) Let Q_{n-2} be an $(n-2) \times (n-2)$ permutation matrix such that the first half of $R_{n-2} = R'_{n-2} Q^t_{n-2}$ is nondecreasing and satisfies the skew-Hankel property.
   (c) Let T_{n-2} be a combinatorially skew-Hankel tournament with score vector R_{n-2} obtained by applying this algorithm, and let T_n be the combinatorially skew-Hankel tournament with score vector R_n defined by

   $$T_n = \left[\begin{array}{c|ccc|c}
   0 & (1-v_{n-1}) & \cdots & (1-v_2) & 0 \\
   \hline
   v_{n-1} & & & & v_2 \\
   \vdots & & Q^t_{n-2} T_{n-2} Q_{n-2} & & \vdots \\
   v_2 & & & & v_{n-1} \\
   \hline
   0 & (1-v_2) & \cdots & (1-v_{n-1}) & 0 \\
   \end{array} \right].$$

(3) Output T_n.

Example 25. Let $n = 7$ and let $R = (2, 2, 4, 2, 4, 2, 2)$. Applying the skew-Hankel algorithm, we obtain the following combinatorially skew-Hankel tournament in $\mathcal{T}_H(R)$:

$$\left[\begin{array}{ccc|c|ccc}
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{array} \right].$$

In carrying out the algorithm, the resulting score vectors are illustrated below where $\to_\pi$ means a permutation is used and $\to_a$ means the result of a step of the algorithm:

$$(2, 2, 4, 2, 4, 2, 2) \to_a (2, 1, 3, 3, 1, 2) \to_\pi (1, 2, 3, 3, 2, 1) \to_a (1, 1, 1, 1) \to_a (0, 0).$$

Theorem 26. *The skew-Hankel algorithm constructs a combinatorially skew-Hankel tournament when R satisfies the given conditions.*

Proof. First assume that n is even. It then suffices to verify that the first half of R'_{n-2} is 2-nearly nondecreasing and that $\sum_{i=1}^{k} r'_i \geq k(k-1)$ for $1 \leq k \leq (n/2) - 1$ (we have equality for $k = (n/2) - 1$ by construction). By construction

$$r'_i = r'_{(n-2)+1-i} = \begin{cases} r_{i+1}, & \text{if } 1 \leq i \leq \lfloor r_1/2 \rfloor, \\ r_{i+1} - 1, & \text{if } \lfloor r_1/2 \rfloor + 1 \leq i \leq \lceil r_1/2 \rceil, \\ r_{i+1} - 2, & \text{if } \lceil r_1/2 \rceil + 1 \leq i \leq (n/2) - 1. \end{cases}$$

The entries of R'_{n-2} are nonnegative: If $r_1 = 0$, then $t_{2,1} = t_{2,n} = 1$ and so $r_2 \geq 2$ and therefore $r'_1 \geq 0$. If $r_1 \geq 1$, then $v_2 = 0$ and thus $r'_1 \geq 0$.

We calculate that

Case $1 \leq k \leq \lfloor r_1/2 \rfloor$:

$$\sum_{i=1}^{k} r'_i = \sum_{i=2}^{k+1} r_i \geq \sum_{i=1}^{k} r_i \geq k(k-1).$$

Case $k = (r_1 + 1)/2$ and r_1 odd, implying that $r_{k+1} \geq r_1 = 2k - 1$:

$$\sum_{i=1}^{k} r'_i = \sum_{i=1}^{k-1} r_{i+1} + (r_{k+1} - 1) \geq (k-1)(k-2) + (2k-1) - 1 = k(k-1).$$

Case $\lceil r_1/2 \rceil + 1 \leq k \leq n/2 - 1$ with r_1 even:

$$\sum_{i=1}^{k} r'_i = \left(\sum_{i=2}^{r_1/2+1} r_i \right) + \left(\sum_{i=r_1/2+2}^{k+1} r_i \right) - 2\left((k+1) - ((r_1/2) + 2) + 1 \right)$$

$$= \left(\sum_{i=2}^{k+1} r_i \right) - 2k + r_1 = \left(\sum_{i=1}^{k+1} r_i \right) - 2k \geq (k+1)k - 2k = k(k-1).$$

Case $\lceil r_1/2 \rceil + 1 \leq k \leq n/2 - 1$ with r_1 odd:

$$\sum_{i=1}^{k} r'_i = \left(\sum_{i=2}^{k+1} r_i \right) - 1 - 2\left((k+1) - ((r_1 + 1)/2 + 2) + 1 \right)$$

$$= \left(\sum_{i=1}^{k+1} r_i \right) - 1 - 2k + 1 \geq (k+1)k - 2k = k(k-1).$$

Since the first half of R is nondecreasing, it follows that the first half of R'_{n-2} is 2-nearly nondecreasing. By Lemma 22 the vector R_{n-2} satisfies the corresponding inequalities. Therefore there is a combinatorially skew-Hankel tournament with score vector R_{n-2} and also one with score vector R'_{n-2}.

Now assume that n is odd. The entries of R'_{n-1} are

$$r'_i = r'_{n-i} = \begin{cases} r_i, & \text{if } 1 \leq i \leq (r_{(n+1)/2})/2, \\ r_i - 1, & \text{if } (r_{(n+1)/2})/2 + 1 \leq i \leq (n-1)/2, \end{cases}$$

and thus the first half of R'_{n-1} is nearly nondecreasing. If $1 \leq k \leq (r_{(n+1)/2})/2$, then $\sum_{i=1}^{k} r'_i \geq k(k-1)$. If $(r_{(n+1)/2})/2 + 1 \leq k \leq (n-1)/2$, then

$$\sum_{i=1}^{k} r'_i = \sum_{i=1}^{k} r_i - (k - (r_{(n+1)/2})/2) \geq k(k-1).$$

Therefore, there exists a combinatorially skew-Hankel tournament with score vector R_{n-1} and also one with score vector R'_{n-1}. $\square$

We next identify certain moves that allow one to move from any combinatorially skew-Hankel tournament $T \in \mathcal{T}_{H^*}(R)$ to any other where each move produces another combinatorially skew-Hankel tournament in $\mathcal{T}_{H^*}(R)$. The switches used in the moves are the 4-cycle switches and 3-cycle switches used in the case of Hankel tournaments, but their types are different. We consider 4-cycle switches $\square_{i,j,n+1-i,n+1-j}$ that reverse the 4-cycle $i \to j \to n+1-i \to n+1-j \to i$, called 4-*cycle skew-Hankel switches*, and pure 3-cycle switches $\triangle_{i,j,k}$ that reverse $i \to j \to k \to i$, that is, where $\{i, j, k\} \cap \{n+1-i, n+1-j, n+1-k\} = \emptyset$ or $\{n/2\}$. A 4-cycle skew-Hankel switch $\square_{i,j,n+1-i,n+1-j}$ is illustrated by

	i	j	$n+1-i$	$n+1-j$
i	0	1		0
j	0	0	1	
$n+1-i$		0	0	1
$n+1-j$	1		0	0

$\to$

	i	j	$n+1-i$	$n+1-j$
i	0	0		1
j	1	0	0	
$n+1-i$		1	0	0
$n+1-j$	0		1	0

.

First assume that n is even. Let $T_1, T_2 \in \mathcal{T}_{H^*}(R)$ and consider T_1 and T_2 as digraphs. For an $n \times n$ combinatorially skew-Hankel tournament, if there is an edge $i \to j$, then there is also an edge $n+1-i \to n+1-j$. Consider the digraph D with vertex set $\{1, 2, \ldots, n\}$ whose edges are the edges of T_1 that are not edges of T_2 (corresponding to the 1s in the difference matrix $T_1 - T_2$). If a vertex has positive outdegree then it has positive indegree, and vice versa. Starting at any vertex, determine a longest path $\gamma : i \to j \to \cdots \to k$ that does not contain both a vertex p and a vertex $n+1-p$. There is an arc leaving vertex k and it must then go either to an earlier vertex on γ or else to a vertex $n+1-p$ where p is a vertex on γ. Thus we either get a cycle

(a) $i_1 \to i_2 \to \cdots \to i_q \to i_1$ where $i_j \neq n+1-i_l$ for any i_j and i_l,

or a cycle

(b) $i_1 \to i_2 \to \cdots \to i_q \to n+1-i_1 \to n+1-i_2 \to \cdots \to n+1-i_q \to i_1$.

In case (a) we also have the cycle

(a′) $n+1-i_1 \to n+1-i_2 \to \cdots \to n+1-i_q \to n+1-i_1$.

Case (a): We show how to reverse the cycle of (a) and of (a′) by a sequence of 3-cycle switches and their complementary switches. There is a sequence of pairs of switches, each consisting of a 3-cycle switch and its complementary switch, which reverses the cycles (a) and (a′) resulting in a combinatorially skew-Hankel tournament in $\mathcal{T}_{H^*}(R)$. We can find the first 3-cycle by considering all the edges of T_1 between i_1 and the other vertices of the cycle (a). Reversing this 3-cycle and its complementary 3-cycle, we obtain one or two shorter cycles. Repeating on these shorter cycles we eventually will reverse cycles (a) and (a′).

Case (b): Consider, for instance, the edges $i_1 \to i_2$ and $n+1-i_1 \to n+1-i_2$. If the edge $n+1-i_2 \to i_1$ is in T_1, and thus so is the edge $i_2 \to n+1-i_1$, then a 4-cycle skew-Hankel switch $\square_{i_1,i_2,n+1-i_1,n+1-i_2}$ yields a tournament in $\mathcal{T}_{H^*}$ where our cycle has become two cycles of the forms (a) and (a′), and we proceed inductively. If the edge $i_1 \to n+1-i_2$ is in T_1, and thus so is the edge $n+1-i_1 \to i_2$, then we have two cycles of the forms (a) and (a′), which inductively by pairs of 3-cycle switches we can reverse, resulting in the reversal of $i_1 \to n+1-i_2$ and $n+1-i_1 \to i_2$. Now a 4-cycle skew-Hankel switch completes the reversal of the cycle of (b) and gives a tournament in $\mathcal{T}_{H^*}(R)$ which is closer to T_2.

For n odd, we have the following lemma.

Lemma 27. *Let T_1 and T_2 be two combinatorially skew-Hankel tournament matrices with the same score vector R. Then there exists a sequence of moves, where each move is a pair consisting of a pure 3-cycle switch $\triangle_{i,j,k}$ and its complementary switch $\triangle_{i,j,k}^c$, which transforms row (respectively, column) $(n+1)/2$ of T_1 into row (respectively, column) $(n+1)/2$ of T_2.*

Proof. As above we consider the digraph D. If row $(n+1)/2$ of T_1 agrees with row $(n+1)/2$ of T_2, there is nothing to prove. Otherwise there is a cycle C containing vertex $(n+1)/2$. We claim there is such a cycle for which if i is a vertex of the cycle, then $(n+1-i)$ is not a vertex of the cycle.

Let C be the cycle $(n+1)/2 \to i_1 \to i_2 \to \cdots \to i_k \to (n+1)/2$. Let $V = \{j : i_j, (n+1-i_j) \in C, 1 \leq j \leq k\}$. If $V \neq \emptyset$, let a be the minimum and c be the maximum of the integers in V. Let b and d be the indices such that $i_b = n+1-i_a$ and $i_d = n+1-i_c$. We consider two cases (see Fig. 2).

Case $b < d$: We consider the cycle C' given by

$$\frac{n+1}{2} \to i_1 \to \cdots \to i_a \to (n+1-i_{b+1}) \to \cdots \to (n+1-i_{d-1}) \to i_c \to \cdots \to i_k \to \frac{n+1}{2}.$$

It follows from the skew-Hankel property that the edges of C' are in D, and for the corresponding V', we have $|V'| < |V|$.

Case $b > d$: We then consider the cycle C' given by

$$\frac{n+1}{2} \to i_1 \to \cdots \to i_a \to \cdots \to i_d \to (n+1-i_{c+1}) \to \cdots \to (n+1-i_k) \to \frac{n+1}{2}.$$
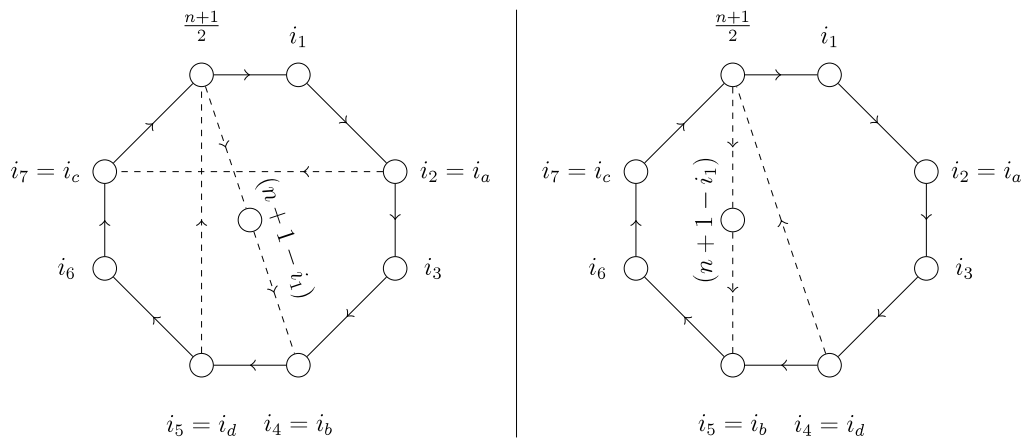
Fig. 2. Two possibilities in Lemma 27 ($b < d$ and $b > d$).

(If $d = a$ and thus $c = b$, $i_a \rightarrow \cdots \rightarrow i_d$ reduces to just the vertex i_a.) As in the above case, we have $|V'| < |V|$. Repeating this process, we eventually produce a cycle C^* such that $V^* = \emptyset$. This verifies our claim.

Now if row and column $(n + 1)/2$ of T_1 do not agree with row and column $(n + 1)/2$ of T_2, we have a cycle C

$$(n + 1)/2 \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k \rightarrow (n + 1)/2$$

in D (and so its reverse is in T_2) such that for each vertex i of C, $(n + 1 - i)$ is not a vertex of C. By the skew-Hankel property, the cycle C^* given by

$$(n + 1)/2 \rightarrow (n + 1 - i_1) \rightarrow (n + 1 - i_2) \rightarrow \cdots \rightarrow (n + 1 - i_k) \rightarrow (n + 1)/2$$

is also in D (and so its reverse is in T_2). Thus we can apply a sequence of moves, each of which is a pair consisting of a 3-cycle switch $\triangle_{i,j,k}$ and its complementary switch $\triangle^c_{i,j,k}$, to T_1 and produce a $T'_1 \in \mathcal{T}_{H^*}(R)$ such that row and column $(n + 1)/2$ of T' are closer to row and column $(n + 1)/2$ of T_2. We continue like this until we get row and column $(n + 1)/2$ of T_2. □

Thus we have proved the following theorem.

Theorem 28. *Let T_1 and T_2 be two combinatorially skew-Hankel tournaments in $\mathcal{T}_{H^*}(R)$. Then there exists a sequence of 4-cycle skew-Hankel switches $\square_{i,j,n+1-i,n+1-j}$ and pairs consisting of a pure 3-cycle switch $\triangle_{i,j,k}$ and its complementary switch $\triangle^c_{i,j,k}$ which brings T_1 to T_2, such that each of the moves produces a combinatorially skew-Hankel tournament in $\mathcal{T}_{H^*}(R)$.*

Example 29. Let $n = 5$ and $R = (1, 2, 2, 2, 1)$, and consider the two combinatorially skew-Hankel tournaments T_1 and T_2 in $\mathcal{T}_{H^*}(R)$ given by

$$T_1 = \left[\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \text{and} \quad T_2 = \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

We can move from T_1 to T_2 by the pair of pure 3-cycle switches $\triangle_{1,3,4}$ and $\triangle_{5,3,2}$.

Again for completeness we mention the following. It is natural to also consider $n \times n$ *combinatorially skew-Hankel loopy tournaments* $T = [t_{ij}]$, that is, combinatorially skew-Hankel tournaments with possible 1s on the main diagonal. By the skew-Hankel property, we now have that $t_{ii} = 1 - t_{n+1-i,n+1-i}$ for all i. The score vector $R = (r_1, r_2, \ldots, r_n)$ of a combinatorially skew-Hankel loopy tournament satisfies $r_i = r_{n+1-i} + 1$ if $t_{ii} = 1$ and $t_{n+1-i,n+1-i} = 0$. Thus the score vector of a combinatorially skew-Hankel loopy tournament determines which elements on the main diagonal equal 1 and which equal 0. If R' is obtained from R by subtracting 1 from those i for which $r_i = r_{n+1-i} + 1$, then there is a combinatorially skew-Hankel loopy tournament with score vector R if and only if there is a combinatorially skew-Hankel tournament with score vector R'.

By taking the columns of a combinatorially skew-Hankel loopy tournament in the reverse order, we obtain a *combinatorially skew-Hankel H-loopy tournament*, that is, a combinatorially skew-Hankel tournament with possible 1s on the Hankel diagonal. Thus combinatorially skew-Hankel H-loopy tournaments are equivalent to combinatorially skew-Hankel loopy tournaments. But we may also consider *combinatorially skew-Hankel doubly-loopy tournaments*, that is, combinatorially skew-Hankel tournaments $T = [t_{ij}]$ with possible 1s on both the main diagonal and the Hankel diagonal. Thus $t_{ii} = 1 - t_{n+1-i,n+1-i}$ for all i and $t_{i,n+1-i} = 1 - t_{n+1-i,i}$ for all i. The score vector $R = (r_1, r_2, \ldots, r_n)$ of a combinatorially

skew-Hankel doubly-loopy tournament satisfies

$$r_i = \begin{cases} r_{n+1-i}, & \text{if } (t_{ii} = 1 \text{ and } t_{i,n+1-i} = 0), \text{ or } (t_{ii} = 0 \text{ and } t_{i,n+1-i} = 1) \\ r_{n+1-i} + 2, & \text{if } t_{ii} = t_{i,n+1-i} = 1 \text{ and so } t_{n+1-i,i} = t_{n+1-i,n+1-i} = 0. \end{cases}$$

Let R' be obtained from R by subtracting 1 from r_i and r_{n+1-i} if $r_i = r_{n+1-i}$ and subtracting 2 from r_i if $r_i = r_{n+1-i} + 2$. Then R is the score vector of a combinatorially skew-Hankel doubly-loopy tournament if and only if R' is the score vector of a combinatorially skew-Hankel tournament. Notice that when $r_i = r_{n+1-i}$ we can either put 1s in positions (i, i) and $(n + 1 - i, i)$ or in positions $(i, n + 1 - i)$ and $(n + 1 - i, n + 1 - i)$.

5. Summary

In this paper we have obtained necessary and sufficient conditions for the existence of loopy tournaments, Hankel tournaments, and combinatorially skew-Hankel tournaments with a prescribed score vector. We have also given algorithms for their construction when these conditions are satisfied. In addition, we have shown how to move from one tournament to another tournament in the same class by moves given by switches and pairs of switches. The moves used in each case are:

(a) loopy tournaments: $\rightarrow \circ_{ij}$; $\triangle_{i,j,k}$
(b) Hankel tournaments: $\triangle_{i,j,k}$ followed by $\triangle^c_{k,j,i}$; $\triangle_{i,(n+1)/2,n+1-i}$; $\square_{i,j,n+1-j,n+1-i}$
(c) combinatorially skew-Hankel tournaments: $\triangle_{i,j,k}$ followed by $\triangle^c_{i,j,k}$; $\square_{i,j,n+1-i,n+1-j}$.

Acknowledgments

References

[1] R.A. Brualdi, Combinatorial Matrix Classes, Cambridge Univ. Press, Cambridge, UK, 2006.
[2] R.A. Brualdi, E. Fritscher, Tournaments associated with multigraphs and a theorem of Hakimi, Discrete Math. 338 (2015) 229–325.
[3] H.G. Landau, On dominance relations and the structure of animal societies. III. The condition for a score structure, Bull. Math. Biophys. 15 (1953) 143–148.
[4] E.S. Mahmoodian, A critical case method of proof in combinatorial mathematics, Bull. Iranian Math. Soc. (1978) 1l–26L.
[5] T.S. Michael, The structure matrix and a generalization of Ryser's maximum term rank formula, Linear Algebra Appl. 145 (1991) 21–31.
[6] K.B. Reid, Tournaments; scores, kings, generalization, and special topics, Congr. Numer. 115 (1996) 171–211.
[7] H.J. Ryser, Matrices of zeros and ones in combinatorial mathematics, in: U. Hans Schndider (Ed.), Recent Advances in Matrix Theory, Wisconsin Press, Madison, 1964, pp. 103–124.
[8] C. Thomassen, Landau's characterization of tournament score sequences, in: The Theory and Applications of Graphs, (Kalamazoo, MI 1980), Wiley, New York, 1981, pp. 589–591.