Linear Preservers and Diagonal Hypergraphs

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Abstract

In this paper we elucidate a connection between linear preservers on coordinate subspaces that permute the elements of a matrix and diagonal hypergraph isomorphisms. Under certain circumstances, the only permutation linear preservers are the special ones which permute rows and columns, and possibly interchange rows with columns as well.

1 Introduction

Let V be a linear space over the complex number field $\mathcal{C}$, and let $\phi : V \to V$ be a linear operator on V. Consider either

(a) a function $f : V \to \mathcal{C}$,

(b) a subset S of V, or

(c) a relation R on V (a subset R of $V \times V$).

The linear operator ϕ is called a *linear preserver* provided, respectively,

(a') $f(\phi(v)) = f(v)$,

(b') $u \in S$ implies $\phi(u) \in S$, or

(c') $(u, w) \in R$ implies $(\phi(u), \phi(w)) \in R$.

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We refer to [7, 11] for recent surveys on linear preservers.

We shall primarily be interested in certain linear spaces of matrices. We denote by $\mathcal{M}_{m,n}(\mathcal{C})$ the linear space of m by n matrices $X = [x_{ij}]$ over $\mathcal{C}$; if $m = n$ we shorten this to $\mathcal{M}_n(\mathcal{C})$. Let $A = [a_{ij}]$ be an m by n matrix of 0's and 1's. Then $\mathcal{M}_A(\mathcal{C})$ is the *coordinate subspace* of $\mathcal{M}_{m,n}(\mathcal{C})$ consisting of all those m by n matrices $Y = [y_{ij}]$ for which $y_{ij} = 0$ for all (i, j) for which $a_{ij} = 0$. We can describe $\mathcal{M}_A(\mathcal{C})$ by using the *Hadamard product* as follows:

$$\mathcal{M}_A(\mathcal{C}) = \{A \circ X = [a_{ij}x_{ij}] : X \in \mathcal{M}_{m,n}(\mathcal{C})\}.$$

For example, if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

then

$$\mathcal{M}_A(C) = \left\{ \begin{bmatrix} a & 0 & b \\ c & d & 0 \\ e & f & g \end{bmatrix} : a, b, c, d, e, f, g \in \mathcal{C} \right\}.$$

If J_{mn} denotes the m by n matrix of all 1's, then $\mathcal{M}_{J_{mn}}(\mathcal{C}) = \mathcal{M}_{m,n}(\mathcal{C})$.

We review here just a few classical results on linear preservers and refer the reader to [7, 11] for more complete information.

In 1897 Frobenius [6], in what is probably the first paper on linear preservers, characterized linear determinant preservers: *A linear operator ϕ on $\mathcal{M}_n(\mathcal{C})$ satisfies*

$$\det \phi(X) = \det X \quad (X \in \mathcal{M}_n(\mathcal{C}))$$

if and only if there exist matrices $P, Q \in \mathcal{M}_n(\mathcal{C})$, with $\det PQ = 1$, *such that ϕ is one of the two linear operators*

$$\phi(X) = PXQ \quad (X \in \mathcal{M}_n(\mathcal{C})), \text{ and } \phi(X) = PX^tQ \quad (X \in \mathcal{M}_n(\mathcal{C})). \tag{1}$$

We say that linear operators on $\mathcal{M}_n(\mathcal{C})$ having one of the forms given in (1) are of *classical type.* Let $E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{n1}, \ldots, E_{nn}$ be the standard ordered basis of $\mathcal{M}_n(\mathcal{C})$. Here E_{ij} is the unit matrix of order n with a 1 in position (i, j) and 0's elsewhere. Then the matrices, with respect to this standard ordered basis, of a linear transformation ϕ on $\mathcal{M}_n(\mathcal{C})$ satisfying (1) equal the tensor products

$$P \otimes Q^t \text{ and } P \otimes Q, \text{ respectively.}$$

A consequence of Frobenius' theorem is that the only linear transformations on $\mathcal{M}_n(\mathcal{C})$ that can used to help compute the determinant are those obtained by applying elementary row and column operations.

The second theorem we wish to mention is due to Marcus and Moyls [8]. It characterizes linear operators that preserve the property of having rank 1. Here we now consider rectangular matrices and *classical type* becomes

$$\phi(X) = PXQ \quad (X \in \mathcal{M}_{m,n}(\mathcal{C})), \text{ and } \phi(X) = PX^tQ \quad (X \in \mathcal{M}_{m,n}(\mathcal{C})), \tag{2}$$

where P and Q are nonsingular matrices of appropriate sizes. *A linear operator* ϕ *on* $\mathcal{M}_{m,n}(\mathcal{C})$ *satisfies*

$$\text{rank } X = 1 \quad \text{implies} \quad \text{rank } \phi(X) = 1, \; (X \in \mathcal{M}_{m,n}(\mathcal{C}))$$

if and only if ϕ *is of classical type.* Note that if ϕ preserves rank 1, then this result implies that ϕ preserves all ranks; hence ϕ is a *strong linear preserver* in the sense that, for each $X \in \mathcal{M}_{m,n}(\mathcal{C})$, $\phi(X)$ has rank 1 if and only if X has rank 1. Minc [10] has given an elementary proof of the theorem of Marcus and Moyls, and derived from it an elementary proof of Frobenius' theorem.

In order for the reader to gain some insight into linear preserver problems (and because we think this is the best way to prove Frobenius' theorem), we outline Minc's deduction of Frobenius' theorem from the theorem of Marcus and Moyls. Let ϕ be a linear operator on $\mathcal{M}_n(\mathcal{C})$ preserving determinant. There are two main steps.

I. ϕ *is invertible*: Suppose that $\phi(A) = O$. There exists a matrix B such that $A + B$ is invertible and the rank of B equals $n -$ rank A. Then

$$\det B = \det \phi(B) = \det(\phi(A) + \phi(B)) = \det \phi(A + B) = \det(A + B) \neq 0.$$

So B is nonsingular, and $A = O$.

II. ϕ *is a rank* 1 *preserver*: Let A be a matrix of rank 1. Then $A = GE_{11}H$ where G and H are nonsingular. Let rank $\phi(A) = k$. By part I., $k \geq 1$. Then $\phi(A) = U(E_{11} + \cdots + E_{kk})V$. Let $B = U(E_{k+1.k+1} + \cdots + E_{nn})V$ for some nonsingular matrices U and V. Then $\det(x\phi(A) + B) = x^k \det UV$, a monomial of degree k. On the other hand,

$$\begin{aligned} \det(x\phi(A) + B) = \det \phi(xA + \phi^{-1}(B)) &= \det(xA + \phi^{-1}(B)) \\ &= \det(G(xE_{11} + G^{-1}\phi^{-1}(B)H^{-1})H)) \\ &= cx. \end{aligned}$$

Hence $k = 1$, that is, the rank of $\phi(A)$ equals 1. Therefore ϕ preserves rank 1 and has classical type. Since ϕ preserves the determinant, $\det PQ = 1$.

Beasley [1] generalized the theorem of Marcus and Moyls by showing that *if* k *is a fixed positive integer with* $k \leq \min\{m, n\}$ *and a linear operator* ϕ *on* $\mathcal{M}_{m,n}(\mathcal{C})$ *preserves matrices of rank* k*, then* ϕ *is of classical type.* There are three corollaries of Beasley's theorem worth noting here:

(a) *If for a fixed positive integer* $k \leq \min\{m, n\}$*, a linear operator* ϕ *on* $\mathcal{M}_{m,n}(\mathcal{C})$ *satisfies rank* $\phi(A) = k$ *whenever rank* $A = k$*, then rank* $\phi(A) =$ *rank* A *for all* A*.*

(b) *A linear operator* ϕ *on* $\mathcal{M}_n(\mathcal{C})$ *preserves nonsingularity if and only if* ϕ *is of classical type.*

(c) *(Extension of the theorem of Frobenius) A linear operator ϕ on $\mathcal{M}_n(\mathcal{C})$ satisfies $|\det \phi(X)| = |\det X|$ for all X if and only if it is of classical type $\phi(X) = PXQ$ or $\phi(X) = PX^tQ$ where $|\det(PQ)| = 1$.*

Assertion (c) follows from (b) since if ϕ preserves the absolute value of the determinant, then it preserves nonsingularity. Finally, we mention that Marcus and Purves [9] proved that *a linear operator ϕ on $\mathcal{M}_n(\mathcal{C})$ preserves the spectrum if and only if it is of classical type where the matrix $Q = P^{-1}$.*

In the next section we consider the special linear operators that rearrange the elements of a matrix.

2 Permutation (linear) operators

There are special combinatorial linear operators ϕ whose action is to permute the entries in the positions of a matrix. Such operators have the property that they determine a bijection on the set of m by n unit matrices and are determined by this bijection: for each unit matrix E_{ij} there is a unique unit matrix E_{kl} such that $\phi(E_{ij}) = E_{kl}$. Such linear operators, called *permutation linear operators*, were first considered in connection with linear preserver problems by Zhan [12]. For example, the linear operator ϕ on $\mathcal{M}_n(\mathcal{C})$ described by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} d & f & g \\ i & a & h \\ b & c & e \end{bmatrix} \tag{3}$$

is a permutation operator where, for instance, $\phi(E_{13}) = E_{32}$ (the element in the $(1,3)$ position moves to the $(3,2)$ position). The linear operators of classical type that are permutation operators are those of the form $\phi(X) = PXQ$ or $\phi(X) = PX^tQ$ where P and Q are permutation matrices of appropriate orders. Such permutation operators are called *special permutation operators*. Zhan [12] used classical results in linear preservers to show that certain permutation linear preservers are special permutation operators. These include determinant preservers, spectrum preservers, and rank preservers.

Our goal is to develop a combinatorial approach that leads to generalizations and some connections with previous studies of a certain hypergraph associated with a square matrix.

3 Combinatorial approach to permutation linear preservers

Let $A = [a_{ij}]$ be a matrix of 0's and 1's of order n, and let

$$V(A) = \{E_{ij} : a_{ij} = 1, 1 \le i, j \le n\}$$

be the set of unit matrices in $\mathcal{M}_A(\mathcal{C})$. We identify such unit matrices E_{ij} with the corresponding positions (i,j) of A, and thus with the positions occupied by 1's in A.

Let ϕ be a permutation operator on $\mathcal{M}_A(\mathcal{C})$ which preserves the absolute value of the determinant:

$$|\det \phi(Y)| = |\det Y| \quad (Y \in \mathcal{M}_A(\mathcal{C})).$$

Let π be a permutation of $\{1, 2, \ldots, n\}$ such that $a_{i\pi(i)} \neq 0$ $(i = 1, 2, \ldots, n)$, and consider a matrix $Y = [y_{ij}]$ of order n such that $y_{ij} \neq 0$ if and only if $\pi(i) = j$. Thus Y is in $\mathcal{M}_A(\mathcal{C})$, the nonzero elements of Y occupy a permutation set of places in Y, and $|\det \phi(Y)| = |\det Y| = |\prod_{i=1}^{n} y_{i\pi(i)}|$. Since ϕ is a permutation operator on $\mathcal{M}_A(\mathcal{C})$, the matrix $\phi(Y) = [y'_{ij}]$ satisfies: there is a permutation ρ of $\{1, 2, \ldots, n\}$ such that (i) $a_{1\rho(1)} = a_{2\rho(2)} = \cdots = a_{n\rho(n)} = 1$, (ii) $y'_{1\rho(1)}, y'_{2\rho(2)}, \ldots, y'_{n\rho(n)}$ are $y_{1\rho(1)}, y_{2\rho(2)}, \ldots, y_{n\rho(n)}$ in some order, and (iii) $y'_{ij} = 0$ if $\rho(i) \neq j$. We can identify ϕ as a bijection from $V(A)$ to itself, and then ϕ sends each permutation set of positions (unit matrices) in $V(A)$ to another permutation set of places.

In [2] the *diagonal hypergraph* $\mathcal{H}(A)$ of the matrix A of 0's and 1's is defined to be the hypergraph with vertex set $V(A)$ whose *hyperedges* are those sets of n vertices of the form

$$f_\pi = \{(i, \pi(i)) : a_{i\pi(i)} = 1, 1 \leq i \leq n\} \tag{4}$$

for some permutation π. A set f_π of positions of A satisfying (4) is called a *diagonal* of A. The hyperedges of $\mathcal{H}(A)$ (the diagonals of A) are in one-to-one correspondence with the *perfect matchings* of the *bipartite graph* $BG(A)$ associated with A in the usual way.[1]

Let $A \circ X = [a_{ij}x_{ij}]$ be a matrix in $\mathcal{M}_A(\mathcal{C})$ where the x_{ij} are complex variables. We sometimes abuse the terminology and refer to the set

$$X_\pi = \{x_{i\pi(i)} : 1 \leq i \leq n, f_\pi \in \mathcal{H}(A)\}$$

as a diagonal of X. Recall that an *automorphism of a hypergraph* is a bijection between its vertices that takes hyperedges to hyperedges. The following lemma is a consequence of our discussion above.

Lemma 3.1 *Let A be a matrix of 0's and 1's and let ϕ be a permutation operator on $\mathcal{M}_A(\mathcal{C})$. If $|\det A \circ X| = |\det \phi(A \circ X)|$ for all X, then ϕ induces an automorphism (also denoted by ϕ) of the diagonal hypergraph $\mathcal{H}(A)$ of A.*

We can think of

$$\det A \circ X = \sum_{\pi : f_\pi \in \mathcal{H}(A)} \pm x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

as a *generating function* (for the hyperedges) of the hypergraph $\mathcal{H}(A)$. The nonzero terms in $\det A \circ X$ correspond bijectively to the hyperedges of $\mathcal{H}(A)$ with each term weighted with 1 or -1 depending on the evenness or oddness of the corresponding permutation. If in place of the determinant we use the permanent, then

$$\text{per } A \circ X = \sum_{\pi : f_\pi \in \mathcal{H}(A)} x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

[1]This bipartite graph has $2n$ vertices, corresponding to the rows and columns of A, and there is an edge between vertices corresponding to row i and column j provided $a_{ij} \neq 0$.

also is a generating function of $\mathcal{H}(A)$.

Again let A be a (0,1)-matrix of order n. A *linear set* of the hypergraph $\mathcal{H}(A)$ is the set of positions of $V(A)$ lying in the same row or column; there are two kinds of linear sets, *row-linear sets* and *column-linear sets*.

First consider the case where A is the matrix J_n of all 1's. Since each pair of positions in $V(J_n)$ lying in different rows and columns is contained in a hyperedge of $\mathcal{H}$, an automorphism of $\mathcal{H}(J_n)$ takes linear sets to linear sets. Since each row-linear set has a nonempty intersection with each column-linear set (and no two row-linear sets nor two column-linear sets intersect), it follows that if an automorphism of $\mathcal{H}(J_n)$ sends one row-linear set to a column linear set then each row-linear set is sent to a column linear set. Thus an automorphism of $\mathcal{H}(J_n)$ either sends row-linear sets to row-linear sets and column-linear sets to column-linear sets, or sends row-linear sets to column-linear sets and column linear sets to row-linear sets. Applying Lemma 3.1 we obtain the following strengthening of the theorem of Zhan [12] already mentioned.

Theorem 3.2 *Let ϕ be a permutation operator on $\mathcal{M}_n(\mathcal{C})$ that preserves the absolute value of the determinant. Then ϕ is a special permutation operator.*

We remark that Theorem 3.2 also follows from corollary (c) of Beasley's theorem and Lemma 1 of [12].

Since the determinant of a matrix is the product of the elements in its spectrum, and since the absolute value of the determinant is the product of its singular values, we obtain the following corollaries.

Corollary 3.3 *Let ϕ be a permutation operator on $\mathcal{M}_n(\mathcal{C})$ that preserves magnitudes of the spectrum. Then ϕ is a special permutation operator of the form $\phi(X) = PAP^{-1}$.*

Proof. From the discussion above, there are permutation matrices P and Q such that $\phi(X) = PXQ$, respectively, $\phi(X) = PX^tQ$, for all $X \in \mathcal{M}_n(\mathcal{C})$. Let D be a diagonal matrix whose diagonal elements have distinct magnitudes. If $\phi(D) = PDQ$ is not a diagonal matrix then $\phi(D)$ has at least two eigenvalues with the same magnitude. Hence $\phi(D)$ is a diagonal matrix, and this implies that ϕ permutes the diagonal elements of each matrix in $\mathcal{M}_n(\mathcal{C})$. We must have $\phi(I_n) = I_n$, and therefore $Q = P^{-1}$. $\diamondsuit$

Corollary 3.4 *Let ϕ be a permutation operator on $\mathcal{M}_n(\mathcal{C})$ that preserves singular values. Then ϕ is a special permutation operator.*

We also remark that if a permutation operator on $\mathcal{M}_n(\mathcal{C})$ preserves rank 1, then it must take linear sets to linear sets. Hence such an operator is a special permutation operator.

4 Linear preservers on coordinate subspaces

Let $A = [a_{ij}]$ be a (0,1)-matrix of order n, with $\mathcal{M}_A(\mathcal{C})$ equal to the coordinate subspace of $\mathcal{M}_n(\mathcal{C})$ determined by A. Suppose there is a position (k, l) such that $a_{kl} = 1$ and there is no diagonal containing the position (k, l). Then for each $Y = [y_{ij}]$ in $\mathcal{M}_A(\mathcal{C})$ and each matrix Y' obtained from Y by replacing y_{kl} with 0, we have $\det Y' = \det Y$. Also if B is the matrix obtained from A by replacing a_{kl} with 0, then $\mathcal{H}(B) = \mathcal{H}(A)$. It is for these reasons that we now assume that for each (k, l) such that $a_{kl} = 1$, there is a diagonal of A containing (k, l); equivalently each vertex of $\mathcal{H}(A)$ belongs to some hyperedge. In standard matrix terminology this is equivalent to saying that A has *total support*. The matrix A is *fully indecomposable* provided it has no r by s zero submatrix with r and s positive integers and $r + s = n$. A fully indecomposable matrix has total support. A matrix with total support is, after applying a special permutation operator, the direct sum of fully indecomposable matrices.

In Theorem 2.4 of [4] it is proved that if two $(0, 1)$-matrices A and B of order n have isomorphic diagonal hypergraphs, then they have the same number of fully indecomposable components; in addition, the fully indecomposable components of A can be paired up with those of B so that corresponding components have isomorphic diagonal hypergraphs. Thus without loss of generality we shall be assuming that A is fully indecomposable. (More information about fully indecomposable matrices and matrices of total support can be found in [5].)

We now extend Theorem 2.4 of [4]. It is more convenient in formulating our theorem (and does not reduce the generality) to assume that the vertices of $\mathcal{H}(A)$ and $\mathcal{H}(B)$ have been labeled in such a way that the isomorphism on the set of labels is the identity. We write $\mathcal{H}(A) \subseteq \mathcal{H}(B)$ whenever the vertices of $\mathcal{H}(A)$ and of $\mathcal{H}(B)$ are assigned labels from the same set in such a way that each hyperedge of $\mathcal{H}(A)$ is a hyperedge of $\mathcal{H}(B)$.

Theorem 4.1 *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be fully indecomposable $(0, 1)$-matrices of order n with the same number of 1's. Assume that $\mathcal{H}(A) \subseteq \mathcal{H}(B)$. Then $\mathcal{H}(A) = \mathcal{H}(B)$.*

Proof. The result is trivial for $n = 1$, and we argue by induction on n. Assume to the contrary that $\mathcal{H}(A)$ is a proper subset of $\mathcal{H}(B)$. There is a vertex x such that the set of hyperedges of $\mathcal{H}(A)$ containing x is a proper subset of the set of hyperedges of $\mathcal{H}(B)$ containing x. Without loss of generality we may assume that A and B have the forms:

$$A = \left[\begin{array}{c|c} x & \\ \hline & A' \end{array}\right] \text{ and } B = \left[\begin{array}{c|c} x & \\ \hline & B' \end{array}\right].$$

The assumptions imply that $\mathcal{H}(A')$ is a proper subset of $\mathcal{H}(B')$. If A' and B' are both fully indecomposable, then we contradict the inductive assumption. So at least one of A' and B' is not fully indecomposable. It follows as in the proof of Theorem 2.4 of [4] that B' must have more fully indecomposable components than A'. We only argue the case where A' is fully indecomposable and B' is not. We consider two cases:

Case 1 : *A' is fully indecomposable and B' has total support.*

Without loss of generality we may assume that

$$B = \left[\begin{array}{c|c|c} x & \alpha & \beta \\ \hline \gamma & B_1' & O \\ \hline \delta & O & B_2' \end{array}\right].$$

Let S be the union of the sets of elements S_α, S_β, S_γ, and S_δ in $\alpha, \beta, \gamma,$ and δ, respectively. Since B is fully indecomposable, each of these sets is nonempty. There is no hyperedge of $\mathcal{H}(B)$ containing x which contains exactly one element from S. Since A' is fully indecomposable, the elements in S cannot belong to A' and hence belong to the first row or first column of A. There is also no hyperedge of $\mathcal{H}(B)$ that contains an element from both S_α and from S_δ, and no hyperedge that contains an element from both S_β and S_γ. It follows that the elements in $S_\alpha \cup S_\delta$ are all contained in either the first row of A or all in the first column of A. Similarly, the elements in $S_\beta \cup S_\gamma$ are all contained in either the first row of A or all in the first column of A. Since there is a hyperedge of $\mathcal{H}(B)$ containing both an element of S_α and of S_γ and one containing both an element of S_β and of S_δ, the elements of $S_\alpha \cup S_\delta$ are in the first row of A, and the elements of $S_\beta \cup S_\gamma$ are in the first column of A (or the other way around). Hence $\mathcal{H}(A)$ has a hyperedge containing elements from both S_α and S_δ, contradicting the facts that $\mathcal{H}(B)$ doesn't and $\mathcal{H}(A) \subseteq \mathcal{H}(B)$.

Case 2 : *A' is fully indecomposable and B' does not have total support.*

Without loss of generality we may now assume that

$$B = \left[\begin{array}{c|c|c} x & \alpha & \beta \\ \hline \gamma & B_1' & O \\ \hline \delta & W & B_2' \end{array}\right]$$

where $W \neq O$. Now we can only assert that the sets S_β and S_γ are nonempty. Let S_W be the nonempty set of elements in W. As in Case 1, the elements in S belong to the first row and first column of A, and arguing as in that case, we may assume that the elements of $S_\alpha \cup S_\delta$ are in the first row of A and those in $S_\beta \cup S_\gamma$ are in the first column of A. Since there is no hyperedge of $\mathcal{H}(B)$ containing an element of both S_α and S_γ, it now follows that $S_\alpha = \emptyset$, and similarly that $S_\delta = \emptyset$. But then, since A is fully indecomposable, an element of S_W must be in the first column of A. Thus $\mathcal{H}(A)$ has a hyperedge containing an element of S_W and an element of exactly one of S_β and S_γ. Since $\mathcal{H}(B)$ contains no such hyperedge, we have a contradiction in this case as well.

It follows that $\mathcal{H}(A) = \mathcal{H}(B)$ and the proof is complete. $\diamondsuit$

We remark that the Theorem 4.1 is not true if we do not assume that both A and B are fully indecomposable. For example, if

$$A = \begin{bmatrix} a & u & 0 & 0 \\ 0 & b & v & 0 \\ 0 & 0 & c & w \\ x & 0 & 0 & d \end{bmatrix} \text{ and } B = \begin{bmatrix} a & u & 0 & 0 \\ v & b & 0 & 0 \\ 0 & 0 & c & w \\ 0 & 0 & x & d \end{bmatrix},$$

then $\mathcal{H}(A)$ is a proper subset of $\mathcal{H}(B)$ and A, but not B, is fully indecomposable.

Unlike the case where $A = J_n$ (see Theorem 3.2), the hypergraph $\mathcal{H}(A)$ in general can have automorphisms that do not result from special permutation operators. Consider the matrix ([5]) defined by

$$A = \begin{bmatrix} a & b & 0 \\ c & d & f \\ 0 & e & g \end{bmatrix} \tag{5}$$

where the letters $a, b, \ldots, g$ denote 1's and at the same time are labels for the positions of the 1's in A (the vertices of $\mathcal{H}(A)$). The hyperedges of $\mathcal{H}(A)$ are: $\{a, d, g\}, \{a, e, f\}, \{b, c, g\}$. The bijection that interchanges b and c and fixes all other vertices of $\mathcal{H}(A)$ is an automorphism of $\mathcal{H}(A)$ which does not result from a special permutation operator. Now let X be the matrix (5) where we think of $a, b, \ldots, g$ as arbitrary complex numbers. Applying this permutation operator to X we get

$$\phi(X) = \begin{bmatrix} a & c & 0 \\ b & d & f \\ 0 & e & g \end{bmatrix}.$$

Obviously, $\det X = \det \phi(X)$. Thus there are non-special permutation operators on coordinate subspaces that preserve the (absolute value of) determinant.

More generally, let A and B be two (0,1)-matrices of order n with the same number of 1's. We now consider permutation operators $\phi : \mathcal{M}_A(\mathcal{C}) \to \mathcal{M}_B(\mathcal{C})$ that preserve that absolute value of the determinant. An obvious generalization of Lemma 3.1 shows that ϕ induces an isomorphism of the diagonal hypergraph of A onto the diagonal hypergraph of B. We state this formally in the next lemma.

Lemma 4.2 *Let A and B be matrices of 0's and 1's and let ϕ be a permutation operator from $\mathcal{M}_A(\mathcal{C})$ to $\mathcal{M}_B(\mathcal{C})$. If $|\det X| = |\det \phi(X)|$ for all X in $\mathcal{M}_A(\mathcal{C})$, then ϕ induces an isomorphism of the diagonal hypergraph $\mathcal{H}(A)$ of A onto the diagonal hypergraph $\mathcal{H}(B)$ of B.*

Proof. The only additional observation needed is to ensure that the hypotheses imply that every hyperedge of $\mathcal{H}(B)$ is the image under ϕ of a hyperedge of $\mathcal{H}(A)$.[2] But if there are n vertices in $\mathcal{H}(A)$ which do not form a hyperedge but are mapped by ϕ to a hyperedge of $\mathcal{H}(B)$, then there is a matrix X in $\mathcal{M}_A(\mathcal{C})$ such that $0 = \det(X) = \det \phi(X) \neq 0$, a contradiction. $\diamondsuit$

If a permutation operator preserves the absolute value of the determinant on a coordinate subspace $\mathcal{M}_A(\mathcal{C})$, then it follows from Lemma 4.2 that it preserves the permanent. We now show that the converse holds as well.

[2]This is automatically satisfied in proving Lemma 3.1 since the number of diagonals cannot change as A is fixed.

Lemma 4.3 *Let A and B be matrices of 0's and 1's and let ϕ be a permutation operator from $\mathcal{M}_A(\mathcal{C})$ to $\mathcal{M}_B(\mathcal{C})$. If* $\operatorname{per} A \circ X = \operatorname{per} \phi(A \circ X)$ *for all X, then* $|\det A \circ X| = |\det \phi(A \circ X)|$ *for all X.*

Proof. Assume that ϕ preserves the permanent of matrices in $\mathcal{M}_A(\mathcal{C})$. By a special permutation operator we can move any diagonal to the main diagonal without changing either the permanent or the absolute value of the determinant. Therefore we can assume that A and B have all 1's on their main diagonals, and that ϕ sends the ith element on the main diagonal of A to the ith element on the main diagonal of B. It then suffices to show that each set C of nonzero positions of A corresponding to a permutation cycle is mapped by ϕ to a set $\phi(C)$ of nonzero positions of B corresponding to a permutation cycle (necessarily of the same length). Since ϕ preserves the permanent, $\phi(C)$ is a union of cycles. If there were more than one cycle in this union, then there is a matrix in $\mathcal{M}_A(\mathcal{C})$ with zero permanent which is mapped by ϕ into a matrix in $\mathcal{M}_B(\mathcal{C})$ with nonzero permanent (consider a cycle in the union and elements on the main diagonal). This contradiction completes the proof. $\diamondsuit$

Isomorphisms of diagonal hypergraphs of matrices were considered in [4]. *If A and B are fully indecomposable matrices, then a diagonal hypergraph isomorphism $\phi : \mathcal{H}(A) \to \mathcal{H}(B)$ that takes linear sets to linear sets is a special permutation operator.*

Another example of a non-special permutation operator ϕ from one coordinate subspace to another that preserves determinant (and so the diagonal hypergraph) is given (with the convention previously introduced) by:

$$A = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & h & i \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & j & 0 \\ 0 & 0 & h & 0 & l \\ 0 & 0 & i & k & m \end{bmatrix}. \quad (6)$$

It is straightforward to check that the formal determinants (and permanents) of A and B (i.e. generating functions of $\mathcal{H}(A)$ and of $\mathcal{H}(B)$) are identical, and thus that (i) $\det X = \det \phi(X)$ for all $X \in \mathcal{M}_A(\mathcal{C})$ and (ii) $\mathcal{H}(A) = \mathcal{H}(B)$. We note that in this isomorphism, the non-linear set $\{b, e, g, h, i\}$ of A becomes a linear set of B, and the linear set $\{b, e, g, j\}$ of A becomes a non-linear set of B.

The matrix A in (6) is obtained by *joining* two square matrices A_1 and A_2 of order 3 with a common element g to produce a matrix whose order is 5, one less that the sum of the orders of the two matrices. We write

$$A = A_1 * A_2, \; A_1 = \begin{bmatrix} 0 & a & b \\ c & d & e \\ f & 0 & g \end{bmatrix}, \; A_2 = \begin{bmatrix} g & h & i \\ j & 0 & k \\ 0 & l & m \end{bmatrix}.$$

(Note that the notation used does not indicate the common element used in the join.) The matrix B in (6) is then the matrix $A_1 * A_2^T$, the matrix obtained from the matrix

A by *partial transposition* of A on A_2. The matrix $C = A_1^T * A_2$ is obtained from A by partial transposition on A_1. The matrix $A_1^T * A_2^T$ is the transpose A^T of A.

The join operation is an associative operation. As a result we may consider successive joins and write $A_1 * A_2 * \cdots * A_k$. Let $A_i' = A_i$ or A^T. Then the matrix $A_1' * A_2' * \cdots * A_k'$ is obtained from $A_1 * A_2 * \cdots * A_k$ by partial transposition. (Note that this implies that A is a partial transposition of itself.) If $A_i' = A_i^T$ for all i, then $A_1' * A_2' * \cdots * A_k' = A^T$. It is straightforward to verify that *partial transposition is a permutation operator from one coordinate subspace to another that preserves the determinant, the permanent, and the diagonal hypergraph.* Partial transpositions can be composed with special permutation operators to produce other permutation operators that preserve the determinant, the permanent, and the diagonal hypergraph.

A set L of elements of A is a *linearizable set* [4] provided there is a matrix B with $\mathcal{H}(A) = \mathcal{H}(B)$ such that L is a linear set of B. Linearizable sets are investigated in [4]. In the next lemma we collect properties of linearizable sets which follow easily from definitions.

Lemma 4.4 *Let A be a fully indecomposable matrix of order n, and let L be a linearizable set of A. Then the following hold:*

(a) $|L| \leq n$;

(b) *For each hyperedge F of $\mathcal{H}(A)$, $|F \cap L| = 1$;*

(c) *L is maximal with respect to the property* (b)*;*

(d) *L is cycle-free (that is, does not contain the set of edges of any cycle of the bipartite graph $BG(A)$ of A);*

(e) *Each cycle of $BG(A)$ intesects L in 0 or 2 elements.*

In hypergraph terminology, properties (b) and (c) are equivalent to the fact that a linear set is a *maximal strongly-stable* set of the hypergraph $\mathcal{H}(A)$. In general, there are non-linear strongly stable sets. The following theorem is proved in [2] (see also [5]).

Theorem 4.5 *Let A be a $(0,1$-matrix of order n with total support. A set S of vertices of $\mathcal{H}(A)$ is a maximal strongly stable set if and only if there exist nonnegative integers p and q with $p + q = n - 1$ and a p by q zero submatrix O_{pq} of A such that S is the set of nonzero positions of A in the complement of O_{pq}: there exist permutation matrices P and Q such that*

$$PAQ = \left[\begin{array}{c|c} O_{pq} & A_2 \\ \hline A_1 & A_{12} \end{array}\right] \qquad (7)$$

where S is the set of nonzero positions of A in the submatrix A_{12}.

If in Theorem 4.5 the set S has cardinality n, then the bipartite graph of A_{12} has $(n-p)+(n-q)=n+1$ vertices and n edges (corresponding to the nonzero positions S); hence if S is in addition cycle-free, then this bipartite graph is a tree. Brualdi and Ross [4] characterized linearizable sets of cardinality n as given in the next theorem. The number of 1's in a (0,1)-vector or matrix x is denoted by $||x||$.

Theorem 4.6 *Let A be a fully indecomposable $(0,1)$-matrix of order $n>1$. Let S be a set of n positive positions of A. There exists a $(0,1)$-matrix B and a permutation operator $\phi : \mathcal{M}_A(\mathcal{C}) \to \mathcal{M}_B(\mathcal{C})$ such that ϕ is an isomorphism from $\mathcal{H}(A)$ onto $\mathcal{H}(B)$ (so ϕ preserves the absolute value of the determinant) that takes S to a linear set in B if and only if there is a p by q zero submatrix O_{pq} of A with $p,q \geq 0$ and $p+q=n-1$ such that after row and column permutations to obtain the form given in (7), the following hold:*

(i) *S is the set of positive positions of the $n-p$ by $n-q$ submatrix A_{12} complementary to O_{pq};*

(ii) *S is cycle-free;*

(iii) *For each row x of A_{12}, the number of rows y of A_2 such that $y \leq x$ (entrywise) equals $||x||-1$;*

(iv) *For each column u of A_{12}, the number of columns v of A_1 such that $v \leq u$ (entrywise) equals $||u||-1$.*

Moreover, every such permutation operator ϕ is a composition of special permutation operators and partial transpositions.

Note that
$$\sum_x (||x||-1) = ||A_{12}|| - (n-p) = n-(n-p) = p,$$
the number of rows of A_2. Since A is fully indecomposable, each row of A_2 contains at least two 1's. If some row y of A_2 satisfied $y \leq x'$ and $y \leq x''$ for two rows x' and x'' of A_{12}, then S would not be cycle-free. Thus when the conditions in Theorem 4.6 hold, each row y of A_2 satisfies $y \leq x$ for exactly one row x of A_{12}.

We now obtain a simpler characterization of linearizable sets of cardinality n.

Theorem 4.7 *Let A be a fully indecomposable $(0,1)$-matrix of order $n>1$. Let S be a set of n positive positions of A. There exists a $(0,1)$-matrix B and a permutation operator $\phi : \mathcal{M}_A(\mathcal{C}) \to \mathcal{M}_B(\mathcal{C})$ such that ϕ is an isomorphism from $\mathcal{H}(A)$ onto $\mathcal{H}(B)$ that takes S to a linear set in B if and only if*

(a) *S is a maximal strongly stable set of $\mathcal{H}(A)$;*

(b) *Each cycle of the bipartite graph of A has 0 or 2 edges in common with S.*

Moreover, every such permutation operator ϕ is a composition of special permutation operators and partial transpositions.

Proof. The theorem asserts the equivalence of conditions (i) to (iv) of Theorem 4.6 with conditions (a) and (b). It follows from Lemma 4.4 that (a) and (b) hold for a linearizable set. Conversely, suppose (a) and (b) hold. Then by Theorem 4.5 there is a p by q zero submatrix O_{pq} of A with $p+q=n-1$ so that (i) and (ii) hold. Without loss of generality we can assume that A has the form given in (7). As already noted,

$$\sum_x(||x||-1) = ||A_{12}|| - (n-p) = p, \tag{8}$$

the number of rows of A_2. Suppose for some row x of A_{12}, the number of rows y of A_2 such that $y \leq x$ is strictly less than $||x||-1$. Then (8) implies that for a different row x' of A_{12} the number of rows y of A_2 with $y \leq x$ is at least equal to $||x'||$. This implies that A has a $||x'||$ by $(q+(n-q-||x'||)) = n-||x'||$ zero submatrix. Since both $||x'||$ and $q+(n-q)-||x'||)$ are positive, this implies that A is not fully indecomposable, a contradiction. Thus (iii), and similarly (iv), of Theorem 4.6 holds. $\diamondsuit$

It has been conjectured in [4] that if for two $(0,1)$-matrices A and B of order n, $\theta : \mathcal{H}(A) \to \mathcal{H}(B)$ is an isomorphism, then θ is a composition of special permutation operators and partial transpositions. We additionally conjecture here that Theorems 4.6 and 4.7 hold without the assumption that S has cardinality n. To prove this conjecture one cannot simply replace a 0 of A with a 1 to get a set S' of cardinality n satisfying the conditions of Theorem 4.7 (or therefore those of Theorem 4.6).

For example, let

$$A = \left[\begin{array}{cccc|c} a & b & 0 & 0 & e \\ 0 & 0 & c & d & f \\ \hline p & q & r & s & 0 \\ 0 & 0 & u & v & 0 \\ x & y & 0 & 0 & 0 \end{array}\right].$$

The set $S = \{a,b,c,d\}$ satisfies (a) and (b) of Theorem 4.7 and there is a unique zero submatrix, the sum of whose dimension equals 4, containing S in its complement. Replacing any 0 in the upper left 2 by 4 submatrix containing S with g, we obtain a matrix A' in which $S' = \{a,b,c,d,g\}$ is a maximal strongly stable set. But the bipartite graph of every such A' contains a cycle intersecting S' in 4 elements.

On the other hand, starting with A and applying special permutation operators and partial transpositons we get:

$$\begin{bmatrix} a & b & 0 & 0 & e \\ 0 & 0 & c & d & f \\ p & q & r & s & 0 \\ 0 & 0 & u & v & 0 \\ x & y & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} a & b & 0 & 0 & e \\ x & y & 0 & 0 & 0 \\ p & q & r & s & 0 \\ 0 & 0 & c & d & f \\ 0 & 0 & u & v & 0 \end{bmatrix} \to \begin{bmatrix} a & b & e & 0 & 0 \\ x & y & 0 & 0 & 0 \\ p & q & 0 & r & s \\ 0 & 0 & f & c & d \\ 0 & 0 & 0 & u & v \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a & b & e & 0 & 0 \\ x & y & 0 & 0 & 0 \\ p & q & 0 & f & 0 \\ 0 & 0 & r & c & u \\ 0 & 0 & s & d & v \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 & e & 0 \\ 0 & 0 & c & r & u \\ 0 & 0 & d & s & v \\ x & y & 0 & 0 & 0 \\ p & q & f & 0 & 0 \end{bmatrix} = A'.$$

Now replacing the 0 in the $(1,3)$ position of A' with g we obtain a set S' of cardinality 5 satisfying (a) and (b). We can now apply Theorem 4.7 and conclude that S is a linearizable set of A and that by special permutation operators and partial transpositions we can from A get to a matrix in which S is a linear set.

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