MATH 240; FINAL EXAM, 150 points, December 16, 2002 (R.A.Brualdi)

## TOTAL SCORE:

Name: TA (circle): Matthew Petro Dilip Raghavan

Disc. (circle) TUES. THURS. TIME:

Part I (12 multiple choice, 84 points, 7 points each)

Part II (5 questions, 66 points)

Part III (1 bonus question, 10 points)

Part I. [84 points: 12 questions, 7 points each] Multiple Choice Questions.

1. Let  $f(x_1, x_2, ..., x_n)$  be a Boolean function of n Boolean variables  $x_1, x_2, ..., x_n$ . The number of elements in the domain of f, the number of elements in the co-domain of f, and the number of such Boolean functions f are given, respectively, by:

- (a)  $n, 2, \text{ and } 2^n$ .
- (b) 2, 2, and  $2^{2n}$ .
- (c)  $2^n$ , 2, and  $2^{2n}$ .
- (d)  $2^n$ , 2, and  $2^{2^n}$ .

2. The value of the sum  $\sum_{i=1}^{n} 5 \cdot 7^{i}$  equals

- (a)  $\frac{5}{6}(7^{n+1}-1)$
- (b)  $\frac{35}{6}(7^n 1)$ .
- (c)  $5 + \frac{7^{n+1}-1}{6}$
- (d) none of the above.

- 3. Which of the following is **NOT** correct?
  - (a) If p is a prime and a and b are integers such that p|ab, then p|a or p|b.
  - (b)  $100n^3 + 4n^2 = O(n^4)$ .
  - (c) 1111001 (base 2) equals 171 (base 8).
  - (d) 21 has a multiplicative inverse mod 63.

- 4. Identify one of the examples below as **NOT** being an example of a recursive definition, calculation, construction, ... or circle (d) that all are recursive.
  - (a) The sequence  $a_0, a_1, a_2, \dots, a_n, \dots$  is defined by  $a_0 = 3, a_1 = 24$ , and  $a_n = a_{n-1} + a_{n-2}$  for  $n \ge 2$ .
  - (b) The set A of all positive integers which are congruent to 6 mod 11 is given by:  $6 \in A$ , and if  $n \in A$ , then  $n + 11 \in A$ .
  - (c) Let a and b are nonnegative integers (not both zero) with  $a \le b$ . Then  $GCD\{a, b\}$  is given by: If a = 0, then  $GCD\{a, b\} := b$ . Else  $GCD\{a, b\} = GCD\{b \mod a, a\}$ .
  - (d) All are recursive.

- 5. There are 40 different cards each with one of the "ranks" 1, 2, ..., 10 and one of the "colors" hearts, diamonds, clubs, and spades. (i.e the non-picture cards of an ordinary deck of cards). The **number of possible different hands of 5 cards** (i.e. a set of 5 cards from the 40 cards) which are "**full houses**" (i.e. three cards of one rank and two cards of a different rank) equals:
  - (a) C(10,3)C(10,2)
  - (b) C(10,2)C(4,3)C(4,2)
  - (c) P(10,2)C(4,3)C(4,2)
  - (d) none of the above

- 6. Identify the **incorrect statement**:
  - (a) If E and F are two events, the  $\Pr(E \cup F) = \Pr(E) + \Pr(F) \Pr(E \cap F)$
  - (b) If A and B are sets, then  $\overline{\overline{A} \cap B} = A \cup \overline{B}$
  - (c) The number of solutions in nonnegative integers of  $x_1 + x_2 + \cdots + x_n = r$  equals C(n+r-1,n)
  - (d) The solution of the recurrence relation  $h_n = 2h_{n-1} + 1 (n \ge 2)$  with initial condition  $h_1 = 1$  is  $h_n = 2^n 1$ .

- 7. Consider a 2 by n board of unit squares and "pieces" which are either **dominoes** (1 by 2 and 2 by 1 pieces consisting of 2 squares in a row) or **squared-dominoes** (2 by 2 pieces) Let  $a_n$  denote the number of different ways to perfectly tile a 2 by n board with dominoes and squared-dominoes. Thus  $a_1 = 1$  and  $a_2 = 3$ . The **recurrence relation** satisfied by  $a_n$  is:
  - (a)  $a_n = a_{n-1} + a_{n-2} + 1$
  - (b)  $a_n = a_{n-1} + 2a_{n-2}$
  - (c)  $a_n = 2a_{n-1} + a_{n-2}$
  - (d) none of the above.

- 8. Let R and S be two relations on a finite set A of integers with n elements. Identify the statements below which are **NOT** correct. In this problem you might circle more than one answer.
  - (a) If R and S are transitive, then  $R \cap S$  is also transitive.
  - (b) If R and S are transitive, then  $R \cup S$  is also transitive.
  - (c) The transitive closure of R is  $R \cup R^2 \cup R^3 \cup \cdots \cup R^n$
  - (d) If R is the relation congruent modulo 20 and S is the relation congruent modulo 8, then  $R \cap S$  is the relation congruent modulo 160.

- 9. Let  $(S, \preceq)$  be a partially ordered finite set. Which of the following is **always correct**.
  - (a)  $(S, \preceq)$  is a totally ordered set.
  - (b) Each pair of distinct elements of S has an upper bound.
  - (c) S contains at least one minimal element.
  - (d) The (Hasse) diagram of  $(S, \preceq)$  when viewed as a graph is a tree.

10. Identify a property below which is **NOT** a property of every tree T with n > 1 vertices, or else say (e) that all are properties of trees.

- (a) T has n-1 edges.
- (b)  $\chi(T) = 2$
- (c) Inserting a new edge into a tree (but no new vertices) creates exactly one simple cycle.
- (d) If we adjoin a new vertex x to T and insert a new edge joining x to a vertex y of T, then the result is also a tree.
- (e) Properties (a), (b), (c), and (d) are properties of every tree with n > 1 vertices

11. Identify the proposition below which is **NOT** a tautology or circle (e) all are tautologies.

- (a)  $(p \to q) \leftrightarrow (\neg q \to \neg p)$
- (b)  $(p \to q) \leftrightarrow (\neg p \lor q)$
- (c)  $\neg(\forall x \ P(x)) \leftrightarrow \exists \ x \neg P(x)$
- (d)  $(\forall x \exists y \ P(x,y)) \leftrightarrow (\exists \ y \forall x P(x,y))$
- (e) All are tautologies.

12. Identify which of the following is  $\mathbf{NOT}$  an identity or say that (e) all are identities.

(a) 
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, (1 \le k \le n)$$

(b) 
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ (n \ge 1)$$

(c) 
$$(x+3y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} 3^k y^k$$
.

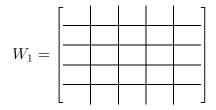
(d) 
$$1+2+3+\cdots+(n-1)=\binom{n}{2}, (n \ge 2)$$

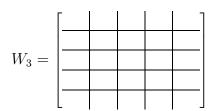
(e) All are identities.

## Part II. (5 questions; 66 points)

13. [12 points] Use Warshall's Algorithm to construct the matrices  $W_1, W_2, W_3$  to PARTLY determine the transitive closure of the relation R on a set of 5 elements, whose matrix representation is given by:

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$





14. [16 points]
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(a) What are the three (defining) properties of an equivalence relation R on a set A?

(b) Consider the partition of  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  into the three sets  $A_1 = \{1, 4, 6, 8\}$ ,  $\{2, 5, 9\}$ , and  $A_2 = \{3, 7\}$  and the equivalence relation R on A determined by it. Determine the equivalence class  $[5]_R$ 

(c) By using partitions, give all possible equivalence relations on the set  $\{1,2,3\}$  of 3 elements?

## 15. [12 points]

(a) Consider the graph shown below with vertices labeled by 1, 2, 3, 4, 5, 6, 7. Use the **Greedy Coloring Algorithm** to color the vertices of the graph using **only** colors 1, 2, 3, 4.

vertex	color
1	
2	
3	
4	
5	
6	
7	

(b) Is the following graph a bipartite graph? If so, label each vertex as L (for Left) or R (for Right).

16. [12 points] **Dijktra's Algorithm** is being used with the following graph with edge weights as shown, in order to find the shortest path from vertex a to every other vertex in the graph. The algorithm has been started in the table shown below, Complete the **next** two lines (and only the next two lines) in the following table.

k	$S_k$	a	b	c	d	e	f	g	h
0	Ø	0	$\infty$						
1	a	0	1	2	4	$\infty$	$\infty$	$\infty$	$\infty$
2	b	0	1	2	4	4	$\infty$	2	$\infty$
3	c	0	1	2	3	4	$\infty$	2	$\infty$
4									
5									
6									
7									
8									

- 17. [14 points]
- (a) The following is the postfix form of an algebraic expression. What is its **value**? (**In the expression all numbers are single digit numbers.**)

$$82 \div 6 \times 645 \times + -5 \times$$

(b) In what order are the vertices visited if  $\mathbf{preorder}$  is used for traversing the following tree?

Part III (10 points, 1 bonus question). Prove that for every positive integer n, 21 divides  $4^{n+1} + 5^{2n-1}$ . Little or no partial credit will be given for this bonus problem.