MATH 240; EXAM # 2, 100 points, November 8, 2002 (R.A.Brualdi)

TOTAL SCORE (10 problems):

Name: Solutions **TA (circle): Matthew Petro Dilip Raghavan**

Disc. (circle) TUES. THURS. TIME:

1. [15 points] **Calculate explicitly** the following:

(a) $P(7,4) = 7!/3! = 840$

(b) $C(8,5) = \binom{8}{5} = 8!/5!3! = 56$

(c) The **coefficient** of x^5y^3 in the expansion of $(2x-y)^8$:
$\binom{8}{3}2^5(-1)^3 = -1792$

(d) The **Boolean matrix product**

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bigodot \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(e) Let S be a program segment, and let p and q be propositions. **Explain the meaning** of the symbol $p\{S\}q$.

If proposition p holds and the program segment S terminates, Then proposition q holds.

2. [12 points] Use **mathematical induction** to prove that

$$3^{2n} \equiv 1 \bmod 8 \text{ for all } n \geq 0.$$

Be sure to specify the two main steps in carrying out the induction.

(1) basis step ($n=0$ in this case): One easily checks that $3^{2\cdot 0} \equiv 1 \bmod 8$.

(2) inductive step: one needs to show that $3^{2n} \equiv 1 \bmod 8$ implies $3^{2(n+1)} \equiv 1 \bmod 8$.

So suppose that $3^{2n} \equiv 1 \bmod 8$. Then $3^{2(n+1)} = 3^{2n}\cdot 3^2 = 3^{2n}\cdot 9 \equiv 1\cdot 1 \equiv 1 \bmod 8$, since $9 \equiv 1 \bmod 8$.

Thus the statement follows by induction.

3. [5 points]
Consider a theorem of the form:

Theorem: If p, then q.

Explain the difference between **a direct proof and an indirect proof** of the Theorem.

In a direct proof, one assumes that p is true and shows that q is true; i.e. the line p true and q false cannot occur in the truth table of the implication.

In an indirect proof, one assumes that q is false, and shows that p is false, leading to the same conclusion.

4. [12 points] Let $s_n = 1^5 + 2^5 + \cdots + n^5$. Give recursive and iterative algorithms for computing s_n.

(i) **Recursive Algorithm**:

If $n = 1$, then $s_1 := 1$.

Else $s_n := s_{n-1} + n^5$.

(ii) **Iterative Algorithm**:

$s_1 := 1$
For $i = 2, 3, \ldots, n$, $s_i := s_{i-1} + i^5$.

5. [12 points] A password is to be 7 characters long with each character equal to one of the 10 digits $0, 1, \ldots, 9$ or one of the 26 lowercase letters $a, b, c, \ldots, x, y, z$. (Repeated characters are allowed.)

(i) **How many** passwords are there such that the **characters are distinct and one of the characters equals 0**?

$7 \cdot 35 \cdot 34 \cdot \cdots \cdot 30$

(ii) If a password is chosen at random from all possible passwords, what is the **probability** its characters are distinct and one of the characters equals 0?

above answer/36^7

6. [12 points] A bakery has 6 different kinds of bagels and has on hand a large number of each kind. You want to buy a bag of 2 dozen bagels containing at least one bagel of each type. **How many different bags could you buy**?

Number of solutions in nonnegative integers of $x_1 + x_2 + \cdots + x_6 = 18$ and so $\binom{18+6-1}{5} = \binom{23}{5}$

7. [10 points] Messages are to be transmitted over a communications channel using three different signals: one requires 1 nanosecond, another requires 2 nanoseconds, and a third requires 4 nanoseconds. A message consists of sequences of these three signals where each signal is immediately followed by the next signal.

(i) Give a **recurrence relation** for the number f_n of different messages that can be sent in exactly n nanoseconds, $n \geq 1$.

$f_n = f_{n-1} + f_{n-2} + f_{n-4}, m \geq 5.$

(ii) What are the **initial conditions** which together with the recurrence relation in (i) determine f_n for all n. **Do NOT solve the recurrence relation.**

$f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 6.$

8. [12 points] Find an **explicit formula** for the general term a_n of the sequence $a_1, a_2, \ldots, a_n, \ldots$ that satisfies:

$$a_n = a_{n-1} + 6a_{n-2}, n \geq 2 \text{ with } a_0 = 3, a_1 = 1.$$

Characteristic equation is $x^2 - x - 6 = 0$ which has roots 3 and -2, giving the general solution $a_n = c3^n + d(-2)^n$. Using the initial conditions we get $c = 7/5$ and $d = 8/5$.

9. [10 points] **How many permutations** are there of the 9 letters Y,O,U,A,R,E,H,I,T that contain at least one of the words YOU, ARE, HIT?

Let A be those permutations that contain YOU, let B be those containing ARE, and let C be those containing HIT. We want $A \cup B \cup C$. Using the Inclusion-Exclusion Principle we get

$3 \cdot 7! - 3 \cdot 5! + 3!.$