Aztec Diamonds and Digraphs, and Hankel Determinants of Schröder Numbers

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Abstract

The Aztec diamond of order n is a certain configuration of $2n(n+1)$ unit squares. We give a new proof of the fact that the number Π_n of tilings of the Aztec diamond of order n with dominoes equals $2^{n(n+1)/2}$. We determine a sign-nonsingular matrix of order $n(n+1)$ whose determinant gives Π_n. We reduce the calculation of this determinant to that of a Hankel matrix of order n whose entries are large Schröder numbers. To calculate that determinant we make use of the J-fraction expansion of the generating function of the Schröder numbers.

1 Introduction

Let n be a positive integer. The *Aztec diamond of order* n is the union AD_n of all the unit squares with integral vertices (x,y) satisfying $|x|+|y| \leq n+1$. The Aztec diamond of order 1 consists of the 4 unit squares which have the origin $(0,0)$ as one of their vertices. The Aztec diamonds of orders 2 and 4 are shown in Figure 1. Aztec diamonds are invariant under rotation by 90 degrees, and by reflections in the horizontal and vertical axes. The part of the Aztec diamond of order n that lies in the positive quadrant consists of a staircase pattern of $n, n-1, \ldots, 1$ unit squares. Thus the Aztec diamond of order n contains

$$4\left(\sum_{i=1}^{n} i\right) = 2n(n+1)$$

unit squares.

The number Π_n of tilings of the Aztec diamond of order n with dominoes is $2^{n(n+1)/2}$ and this was first calculated in [6, 7], with four proofs given. Other calculations of these tilings are given in [4], [12], and [2]. Ciucu [4] derives the recursive relation $\Pi_n = 2^n\Pi_{n-1}, n \geq 2$ which, with $\pi_1 = 2$, immediately gives $\Pi_n = 2^{n(n+1)/2}$. Kuo [12] used a method he called graphical condensation (inspired by a classical determinant technique of Dodgson [5] called condensation), to derive the recursion

$$\Pi_n\Pi_{n-2} = 2\Pi_{n-1}^2,\ (n \geq 3),$$

from which, with $\Pi_1 = 2$ and $\Pi_2 = 8$, the formula for Π_n also follows immediately. In [8] the number of tilings of Aztec diamonds with defects are counted. Additional references on these and related questions can be found in the references cited here.

Tilings of Aztec diamonds are in one-to-one correspondence with the perfect matchings of Aztec graphs. First recall that a *perfect matching* in a graph is a collection Θ of edges such that each vertex of the graph is a vertex of exactly one edge in Θ. The *Aztec graph* AG_n corresponding to the Aztec diamond AD_n is the graph whose vertices are the squares of the Aztec diamond with two squares joined by an edge if and only if they share a side (and so can be covered by one domino). The number of vertices of AG_n equals the number of squares $2n(n+1)$ of AD_n, and thus AG_n is a graph of order $2n(n+1)$. A drawing of the graph AG_n can be obtained from a drawing of AD_n by taking the centers of the squares of AD_n as the vertices and joining two centers by a line segment provided the corresponding squares share a side. The graph AG_4 is obtained from AD_4 in this way in Figure 2.

Using the black-white checkerboard coloring of AD_n, we see that the Aztec graph AG_n is a bipartite graph (let the vertex take the color of the square containing it). A bi-adjacency matrix B_n of AG_n (formed by choosing an ordering of the black vertices and an ordering of the white vertices) is an $n(n+1)$ by $n(n+1)$ $(0,1)$-matrix and completely characterizes AG_n and AD_n. The perfect matchings of AG_n are in one-to-one correspondence with the permutation matrices P satisfying $P \leq B_n$, where the inequality is entrywise. Thus the number Π_n of tilings of AD_n equals the *permanent* of B_n defined by

$$\text{per}(B_n) = \sum_{\sigma} \prod_{i=1}^{n(n+1)} b_{i\sigma(i)},$$

where the summation extends over all permutations σ of $\{1, 2, \ldots, n(n+1)\}$. Hence $\text{per}(B_n) = 2^{n(n+1)/2}$.

Let G be a bipartite graph with bipartition $\{X, Y\}$ having a perfect matching Θ. Associated with G and each choice of Θ is a digraph $D(G, \Theta)$. Let $\Theta = \{m_1, m_2, \ldots, m_p\}$, where $m_i = \{x_i, y_i\}$, $(i = 1, 2, \ldots, p)$ and $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$. The vertices of $D(G, \Theta)$ are $m_1, m_2, \ldots, m_p$, and there is an arc from $m_r = \{x_r, y_r\}$ to $m_s = \{x_s, y_s\}$ in $D(G, \Theta)$ if and only $r \neq s$ and there is an edge $\{x_r, y_s\}$ in G joining x_r and y_s. Let B be the bi-adjacency matrix of G with rows corresponding, in order, to $x_1, x_2, \ldots, x_p$ and columns corresponding, in order, to $y_1, y_2, \ldots, y_p$. The elements on the main diagonal of B all equal 1, and the matrix $B - I_n$ is the adjacency matrix of the digraph $D(G, \Theta)$.[1] The number of perfect matchings of G, the permanent of B, is the same as the number of collections of pairwise vertex disjoint directed cycles of $D(G, \Theta)$.[2] This follows since any collection of pairwise disjoint directed cycles of $D(G, \Theta)$ corresponds to a permutation matrix P' contained in some principal submatrix B' of B, and P' can be uniquely extended to a permutation matrix P using the 1's on the main diagonal of the complementary submatrix B'' of B' in B.

A square $(0, 1, -1)$-matrix is *sign-nonsingular*, abbreviated as *SNS*, provided there is a nonzero term in its standard determinant expansion and all nonzero terms have the same sign. Let $B = [b_{ij}]$ be a $(0, 1)$-matrix of order p with a nonzero term in its standard determinant expansion, and suppose it is possible to replace some of its 1's with -1's in order to obtain an SNS-matrix $\widehat{B} = [\hat{b}_{ij}]$. We call $\widehat{B}$ an *SNS-signing* of B. It follows that

$$|\det(\widehat{B})| = \mathrm{per}(B).$$

Thus $\mathrm{per}(B)$ can be computed using a determinant calculation. The advantage is that, unlike the permanent, there are efficient algorithms to calculate the determinant. This idea was used by Kastelyn [9, 10] in solving the dimer problem of statistical mechanics (see [3] for history and a thorough development of SNS-matrices). Assume that $I_p \leq B$ and, without loss of generality, that $\widehat{B}$ has all -1's on its main diagonal.[3] Since the product of the main diagonal elements equals $(-1)^p$, $\widehat{B}$ is an SNS-matrix if and only if all the nonzero terms in its standard determinant expansion have sign $(-1)^p$. Let $D(\widehat{B})$ be the *signed digraph* of $\widehat{B}$ with vertices $\{1, 2, \ldots, p\}$ and an arc from i to j of sign $\hat{b}_{ij}$ provided $i \neq j$ and $\hat{b}_{ij} \neq 0$. Define the *sign of a directed cycle* to be the product of the signs of its arcs. The theorem of Bassett, Maybee, and Quirk [1] asserts, and an elementary calculation shows [3], that $\widehat{B}$ is an SNS-matrix if and only if the sign of every directed cycle of $D(\widehat{B})$ is -1.

In this paper we consider a specific perfect matching Θ of the bipartite graph AG_n leading to a digraph which we call an *Aztec digraph*. We then determine an SNS-signing $\widehat{B_n}$ of the associated bi-adjacency matrix B_n. We evaluate the determinant of $\widehat{B_n}$ by using the technique of the Schur complement, with respect to a strategically chosen principal submatrix. This leads to a matrix whose elements are the (large) Schröder numbers, and then to the computation of a Hankel determinant of order n (in contrast to the order $n(n+1)$ of $\widehat{B_n}$. We then use a J-fraction expansion to calculate the Hankel determinant. The result is a new and interesting proof that the number of tilings of the Aztec diamond

[1]We emphasize that $D(G, \Theta)$ and B^* depend on the choice of perfect matching Θ.

[2]We include here the empty collection of directed cycles which corresponds to the perfect matching Θ, equivalently, in the permanent calculation, to $I_n \leq B$.

[3]This can be accomplished first by multiplication on the left by a permutation matrix and then by multiplication by a diagonal matrix whose main diagonal elements are 1 or -1, without affecting the sign-nonsingularity property.

AD_n equals $2^{n(n+1)/2}$. Further, the proof's technique is, we think, potentially transferable to similar combinatorial problems.

2 The Aztec Digraph and SNS-Matrix

Let n be a positive integer. We define the *dual-Aztec diamond* of order n to be the union AD_n^d of all the unit squares with vertices $(1/2)(x,y)$ where x and y are *odd* integers satisfying $|x|+|y| \leq 2n$. The centers of the squares of the Aztec diamond of order n are the vertices of the squares of the dual-Aztec diamond of order n. The Aztec graph AG_n of order n can be identified as the vertex-edge graph of the dual-Aztec diamond AG_n^d, that is, the vertices of AG_n are the vertices of the squares of AD_n^d and the edges are the sides of its squares. Thus Π_n equals the number of perfect matchings of the bipartite graph AG_n as realized in this way. We may further identify the vertices of AG_n as the set

$$V_n = \{(x,y) : x \text{ and } y \text{ odd integers satisfying } |x|+|y| \leq 2n\},$$

and the edges as the set

$$E_n = E'_n \cup E''_n,$$

where

$$E'_n = \{\{(x,y),(x,v)\} : (x,y),(x,v) \in V_n, |y-v| = 2\}$$

and

$$E''_n = \{\{(x,y),(u,y)\} : (x,y),(u,y) \in V_n, |x-u| = 2\}.$$

Consider the subset $\Theta_n^{(*)}$ of E''_n defined by

$$\Theta_n^{(*)} = \cup\{\Theta_n^{(y)} : y = \pm 1, \pm 3, \ldots, \pm(2n-1)\} \tag{1}$$

where for $y = \pm 1, \pm 3, \ldots, \pm(2n-1)$,

$$\Theta_n^{(y)} = \{\{(x,y),(x+2,y)\} : x = -(2n-|y|), -(2n-|y|-4), \ldots, (2n-|y|-6), (2n-|y|-2)\}. \tag{2}$$

Then the edges of $\Theta_n^{(*)}$ constitute a perfect matching of AG_n, and we call $\Theta_n^{(*)}$ the *Aztec matching of order n*. We partition $\Theta_n^{(*)}$ into three sets

$$\begin{aligned}
\Theta_n^{(\pm 1)} &= \Theta_n^{(1)} \cup \Theta_n^{(-1)}, \\
\Theta_n^{(+)} &= \Theta_n^{(3)} \cup \Theta_n^{(5)} \cup \cdots \cup \Theta_n^{(2n-1)}, \text{ and} \\
\Theta_n^{(-)} &= \Theta_n^{(-3)} \cup \Theta_n^{(-5)} \cup \cdots \cup \Theta_n^{(-2n+1)}.
\end{aligned}$$

The *Aztec digraph* of order n is defined to be the digraph $D(AG_n, \Theta_n^{(*)})$ with vertex set $\Theta_n^{(*)}$. The Aztec digraph of order 4 is pictured in Figure 5, as it is obtained from the Aztec graph of order 4 and the Aztec matching $\Theta_4^{(*)}$. There is a natural partition of the arcs of AD_n which is clear from the picture of AD_4 given in Figure 5. There are n two-way arcs (so $2n$ arcs) which are pictured vertically; we refer to these arcs as the *north-south arcs*, and sometimes distinguish them as *North* and *South*. Above these there are arcs which go East, NorthEast, and SouthEast; we refer to these arcs as the *easterly arcs*. Below are the arcs which go West, NorthWest, and SouthWest; we refer to these arcs as the *westerly arcs*. There are no directed cycles made up entirely of easterly arcs and none made up entirely of westerly arcs. Thus every directed cycle uses at least two north-south arcs. In fact, it is easy to see that each directed cycle uses exactly one North arc and exactly one South arc. Hence if we give the sign -1 to the North arcs and the sign $+1$ to every other arc, then the sign of each directed cycle of AD_n is -1. This gives an SNS-signing $\widehat{B_n}$ of the bi-adjacency matrix B_n of the Aztec diamond of order n corresponding to the Aztec matching Θ_n^*. Hence $\widehat{B_n}$ is an SNS-matrix which we call the *nth order Aztec SNS-matrix.*[4] Corresponding to the partition $\Theta_n^{(*)} = \Theta_n^{(\pm 1)} \cup \Theta_n^{(+)} \cup \Theta_n^{(-)}$, there are three induced subdigraphs $D(AG_n, \Theta_n^{(\pm 1)})$ $D(AG_n, \Theta_n^{(+)})$ $D(AG_n, M\Theta_n^{(-)})$, with the latter two subdigraphs acyclic and isomorphic.

Using our notation, we partially summarize as follows.

Theorem 2.1 *For each $n \geq 1$, $\Pi_n = (-1)^{n(n+1)} \det(\widehat{B_n}) = \det(\widehat{B_n})$.* ♣

While our definition determines the Aztec digraph, we need to choose a particular ordering of the matching edges in $\Theta_n^{(*)}$ in order to uniquely specify its bi-adjacency matrix $\mathcal{A}_n$.[5]

We now specify an ordering of the matching edges in $\Theta_n^{(*)}$. We first take the edges in Θ_n^{-1} in the order reverse of that specified by (2) and then the edges in Θ_n^1 again in the order reverse of that specified by (2). The edges in $\Theta_n^{(-)}$ come next followed by the edges in $\Theta_n^{(+)}$. It remains to specify an ordering for the edges in these two sets, and we do this next. First consider $\Theta_n^{(-)}$. We consider the natural order of the edges in each Θ_n^y as specified in (2) by increasing values of x. The edges in $\Theta_n^{(-)}$ are in a triangular formation according to the values of $y = -3, -5, \ldots, -(2n-1)$. We select them in the order: last edge in Θ_n^{-3}, last edge of Θ_n^{-5}, second-from-last edge in Θ_n^{-3}, last edge in Θ_n^{-7}, second-from-last edge in Θ_n^{-5}, third-from-last edge in Θ_n^{-3}, last edge in Θ_n^{-9}, etc. We specify an ordering for the edges in the set $\Theta_n^{(+)}$ in a similar way. In Figure 5, the edges of the Aztec matching are labeled from 1 to 20 according to the prescription given. With this labeling, the SNS-matrix $\widehat{B_4}$ is given by:

[4] $\widehat{B_n}$ is a matrix of order $n(n+1)$.

[5] Otherwise, the bi-adjacency matrix is only determined up to permutation similarity, that is, $P\mathcal{A}_nP^T$ where P is a permutation matrix.

$$\left[\begin{array}{rrrr|rrrr|rrrrrr|rrrrrr}
-1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right].$$

Let $N = \binom{n}{2}$. For $n \geq 2$, let P_n be the $(0,1)$-matrix of order n which has 1's on its superdiagonal and 0's elsewhere, let Q_n be the back-diagonal permutation matrix of order n (with 1's in positions $(1,n), (2,n-1), \ldots, (n,1)$ and 0's elsewhere), and let M_N denote an upper-triangular $(0,1,-1)$-matrix of order N with -1's on the diagonal and 0's and 1's off the main diagonal in certain positions. Also let $X_{n,N}$ and $Y_{N,n}$ denote certain $(0,1)$-matrices of sizes n by N and N by n, respectively. Then the nth order Aztec SNS-matrix has the form:

$$\widehat{B_n} = \left[\begin{array}{c|c|c|c}
-I_n + P_n & -I_n & X_{n,N} & O_{n,N} \\
\hline
I_n & -I_n + P_n^T & O_{n,N} & Q_n X_{n,N} \\
\hline
Y_{N,n} & O_{N,n} & M_N & O_N \\
\hline
O_{N,n} & Y_{N,n} Q_n & O_N & M_N
\end{array}\right]. \tag{3}$$

Here P_n corresponds to the East arcs in the subdigraph $D(AG_n, \Theta_n^{(\pm 1)})$ while P_n^T corresponds to the West arcs in this subdigraph. The third diagonal block M_N equals $-I_N + U_N$ where U_N is the adjacency matrix of $D(AG_n, \Theta_n^{(-)})$, and the fourth diagonal block M_N equals $-I_N + U_n$ where U_N is also the adjacency matrix of $D(AG_n, \Theta_n^{(+)})$. The matrices $X_{n,N}$ and $Y_{N,n}$ correspond to the arcs from $\Theta_n^{(-1)}$ to $\Theta_n^{(-)}$ and from $\Theta_n^{(-)}$ to $\Theta_n^{(-1)}$, respectively. The matrices $Q_n X_{n,N}$ and $Y_{N,n} Q_n$ correspond in a similar way to the arcs between $\Theta_n^{(1)}$ and $\Theta_n^{(+)}$.

3 Schur Complements and Schröder Numbers

We begin by recalling the idea of a Schur complement and the resulting Schur determinant formula.

Let A be a matrix of order n partitioned as in

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$$

where A_1 is a nonsingular matrix of order k. Let

$$C = \begin{bmatrix} I_k & O \\ -A_{21}A_1^{-1} & I_{n-k} \end{bmatrix}.$$

Then

$$CA = \begin{bmatrix} A_1 & A_{12} \\ O & A_2 - A_{21}A_1^{-1}A_{12} \end{bmatrix},$$

Since $\det(C) = 1$, it follows that

$$\det(A) = \det(A_1)\det(A_2 - A_{21}A_1^{-1}A_{12}). \tag{4}$$

The matrix $A_2 - A_{21}A_1^{-1}A_{12}$ is called the *Schur complement* of A_1 in A, and the determinant formula (4) is *Schur's formula*. As seen by our calculation, the Schur complement results by adding linear combinations of the first k rows of A to the last $n-k$ rows.

Next we recall the sequence of (*large*) *Schröder numbers* $(r(n) : n \geq 0)$ which begins as

$$1, 2, 6, 22, 90, 394, 1806, \ldots.$$

The Schröder number $r(n)$ is defined to be the number of lattice paths in the xy-plane which start at $(0,0)$, end at (n,n), and use horizontal steps $(1,0)$, vertical steps $(0,1)$, and diagonal steps $(1,1)$, and never pass above the line $y = x$. Such paths are often called *Schröder paths*. The sequence $(s(n) : n \geq 1)$ of (*small*) *Schröder numbers* begins as

$$1, 1, 3, 11, 45, 197, 903, \ldots.$$

We have

$$r(n) = 2s(n+1) \text{ for } n \geq 1 \text{ with } r(0) = 1. \tag{5}$$

The generating function for the small Schröder numbers $s(n)$ is

$$\sum_{n=1}^{\infty} s(n)x^n = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4},$$

and they satisfy the recursive formula

$$(n+1)s(n+1) - 3(2n-1)s(n) + (n-2)s(n-1) = 0 \ (n \geq 2), \ \ s(1) = 1, s(2) = 1.$$

The large Schröder numbers then satisfy

$$(n+3)r(n+2) - 3(2n+3)r(n+1) + nr(n) = 0 \ (n \geq 0), \ \ r(0) = 1, r(1) = 2,$$

and it follows from (5) that their generating function is

$$\sum_{n=0}^{\infty} r(n)x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

For these relationships and other combinatorial interpretations of Schröder numbers, one may consult [13, 14, 15].

4 Schur Complementation of the Aztec SNS-matrix

Consider the nth order Aztec SNS-matrix $\widehat{B_n}$ and its principal, nonsingular submatrix $M_N \oplus M_N$. Taking the Schur complement of $M_N \oplus M_N$ in $\widehat{B_n}$ and using Schur's determinant formula, we get that

$$\det(\widehat{B_n}) = \det(M_N)^2 \det \begin{bmatrix} E_n & -I_n \\ I_n & F_n \end{bmatrix} = \det \begin{bmatrix} E_n & -I_n \\ I_n & F_n \end{bmatrix}. \tag{6}$$

where

$$E_n = -I_n + P_n - X_{n,N} M_N^{-1} Y_{N,n}, \tag{7}$$

and

$$F_n = -I_n + P_n^T - Q_n X_{n,N} M_N^{-1} Y_{N,n} Q_n. \tag{8}$$

Recall that a *Toeplitz matrix* $T(c_{-(n-1)}, \ldots, c_{-1}, c_0, c_1, \ldots, c_{n-1})$ is a matrix $T = [t_{ij}]$ of order n such that $t_{ij} = c_{j-i}$ for $i, j = 1, 2, \ldots, n$. For example,

$$T(c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3) = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_{-1} & c_0 & c_1 & c_2 \\ c_{-2} & c_{-1} & c_0 & c_1 \\ c_{-3} & c_{-2} & c_{-1} & c_0 \end{bmatrix}.$$

Lemma 4.1 *For $n \geq 0$, the matrix F_n is the lower triangular Toeplitz matrix*

$$T(r(n-2), \ldots, r(2), r(1), r(0)+1, -1, 0, 0, \ldots, 0)$$

of order n, where $r(0), r(1), \ldots, r(n-1)$ are large Schröder numbers. The matrix E_n equals

$$F_n^T = T(0, 0, \ldots, 0, -1, r(0)+1, r(1), r(2), \ldots, r(n-2)).$$

Proof. First we consider the matrix $M_N = -I_N + U_N = -(I_N - U_N)$ where U_N is the adjacency matrix of $D(AG_n, \Theta_n^{(-)})$. Since U_N is a strictly upper triangular matrix (so a nilpotent matrix) that records the arcs from $\Theta_n^{(-)}$ to $\Theta_n^{(-)}$, we have

$$M_N^{-1} = (I_N - U_N)^{-1} = -\left(I_N + U_N + U_N^2 + \cdots + U_N^{N-1}\right).$$

Hence the element of M_N in position (k, l) is 0 if $k > l$, -1 if $k = l$, and the number of paths in $D(AG_n, \Theta_n^{(-)})$ from its kth vertex to its lth vertex if $k < l$. Since $X_{n,N}$ records the arcs from $\Theta_n^{(-1)}$ to $\Theta_n^{(-)}$ and $Y_{N,n}$ records the arcs that go the other way, it follows that

$$X_{n,N} M_N^{-1} Y_{N,n}$$

records the number of paths from the ith vertex of $\Theta_n^{(-1)}$ to its jth vertex. This number is 0 if $j \leq i$ and equals the kth Schröder number $r(k)$ if $j > i$ and $k = j - i - 1$. Multiplying on the left and right by the back-diagonal matrix Q_n reorders the rows and columns from last to first. The matrix P_n^T has 1's in the subdiagonal and 0's elsewhere. Adding $-I_n + P_n^T$, we get the Toeplitz matrix $F_n = T(r(n-2), \ldots, r(2), r(1), r(0)+1, -1, 0, 0, \ldots, 0)$. That $E_n = F_n^T$ follows by symmetry. ♣

For the case $n = 4$, corresponding to Figure 5, the Toeplitz matrix F_4 in Lemma 4.1 is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & 2 & -1 & 0 \\ 6 & 2 & 2 & -1 \end{bmatrix}.$$

By (6) and Lemma 4.1, we have reduced the calculation of the determinant of the SNS-matrix $\widehat{B_n}$ of order $n(n+1)$ to the calculation of a determinant of a matrix of order $2n$:

$$\det(\widehat{B_n}) = \det \begin{bmatrix} F_n^T & -I_n \\ I_n & F_n \end{bmatrix}.$$

We further reduce the calculation to the determinant of a matrix of order n:

$$\det(\widehat{B_n}) = \det \begin{bmatrix} F_n^T & -I_n \\ I_n & F_n \end{bmatrix} \;=\; \det \begin{bmatrix} I_n & O_n \\ -(F_n^T)^{-1} & I_n \end{bmatrix} \det \begin{bmatrix} F_n^T & -I_n \\ I_n & F_n \end{bmatrix}$$

$$
\begin{aligned}
&= \det \begin{bmatrix} F_n^T & -I_n \\ O_n & F_n + (F_n^T)^{-1} \end{bmatrix} \\
&= \det(F_n^T) \det\left(F_n + (F_n^T)^{-1}\right) \\
&= (-1)^n \det\left(F_n + (F_n^{-1})^T\right).
\end{aligned}
$$

In order to evaluate this last determinant, we need to compute F_n^{-1}. To do this we first derive a recurrence relation for the Schröder numbers $r(n)$.

Lemma 4.2 *The Schröder numbers $(r(n) : n \geq 0)$ satisfy*

$$
r(n) = r(n-1) + \sum_{k=0}^{n-1} r(k) r(n-1-k) \text{ for } n \geq 1, \text{ with } r(0) = 1.
$$

Proof. The Schröder number $r(n)$ equals the number of lattice paths γ that begin at $(0,0)$ and end at (n,n) which use steps of the type $(1,0)$, $(0,1)$, and $(1,1)$, and never pass above the line $y = x$. There are $r(n-1)$ such paths γ_1 that begin with the diagonal step $(1,1)$. The remaining paths γ_2 begin with the horizontal step $(1,0)$. There is a first value of x between 1 and n such that a path γ_2 crosses the line $y = x-1$, necessarily by a vertical step $(0,1)$. The number of such paths γ_2^k that cross at $x = k$ equals $r(k-1)r(n-1-(k-1))$. Hence

$$
\begin{aligned}
|\{\gamma\}| &= |\{\gamma_1\}| + \sum_{k=1}^{n} |\{\gamma_2^k\}| \\
&= r(n-1) + \sum_{k=1}^{n} r(k-1) r(n-1-(k-1)) \\
&= r(n-1) + \sum_{k=0}^{n-1} r(k) r(n-1-k).
\end{aligned}
$$

♣

Lemma 4.3 *For $n \geq 2$, the inverse of the Toeplitz matrix*

$$
F_n = T(r(n-2), \ldots, r(2), r(1), r(0)+1, -1, 0, 0, \ldots, 0)
$$

of order n is the Toeplitz matrix

$$
-T(r(n-1), \ldots, r(2), r(1), r(0), 0, 0, \ldots, 0) = T(-r(n-1), \ldots, -r(2), -r(1), -r(0), 0, 0, \ldots, 0).
$$

Proof. We prove the lemma by induction on n. The relation is true for $n = 2$ since

$$\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}.$$

We now proceed by induction assuming the relation holds for some $n \geq 2$. We have that

$$F_{n+1} = \left[\begin{array}{cccc|c} & & & & 0 \\ & & F_n & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline r(n-1) & \cdots & r(1) & r(0)+1 & -1 \end{array}\right]$$

and also

$$F_{n+1} = \left[\begin{array}{c|cccc} -1 & 0 & 0 & \cdots & 0 \\ \hline r(0)+1 & & & & \\ r(1) & & & F_n & \\ \vdots & & & & \\ r(n-1) & & & & \end{array}\right].$$

Computing the inverses of F_{n+1} using each of these forms, we get

$$F_{n+1}^{-1} = \left[\begin{array}{c|c} & 0 \\ F_n^{-1} & 0 \\ & \vdots \\ & 0 \\ \hline x^T & -1 \end{array}\right], \tag{9}$$

and

$$F_{n+1}^{-1} = \left[\begin{array}{c|cccc} -1 & 0 & 0 & \cdots & 0 \\ \hline y & & & F_n^{-1} & \end{array}\right], \tag{10}$$

where x and y are vectors of size n. Two applications of the inductive assumption now imply that we need only show that the element α of F_{n+1}^{-1} in position $(n+1, 1)$ equals the Schröder number $r(n)$. Since $F_{n+1}^{-1}F_{n+1} = I_{n+1}$ and $x^T = (\alpha, r(n-1), \ldots, r(1), r(0))$, we have

$$\begin{aligned} \alpha &= r(n-1)(r(0)+1) + r(n-2)r(1) + \cdots + r(1)r(n-2) + r(0)r(n-1) \\ &= r(n-1) + \sum_{k=0}^{n-1} r(k)r(n-1-k). \end{aligned} \tag{11}$$

By Lemma 4.2, $\alpha = r(n)$. ♣

We now have that $\det(\widehat{B_n}) = (-1)^n \det\left(F_n + (F_n^{-1})^T\right)$ where $F_n + (F_n^{-1})^T$ equals the sum of two Toeplitz matrices:

$$T(r(n-2),\ldots,r(2),r(1),r(0)+1,-1,0,0,\ldots,0)+T(0,0,\ldots,0,-r(0),-r(1),-r(2),\ldots,-r(n-1)),$$

and hence equals the Toeplitz matrix

$$T(r(n-2),\ldots,r(2),r(1),r(0)+1,-r(0)-1,-r(1),-r(2),\ldots,-r(n-1)). \tag{12}$$

For example, when $n = 4$ the matrix (12) whose determinant we need to calculate is

$$\begin{bmatrix} -2 & -2 & -6 & -22 \\ 2 & -2 & -2 & -6 \\ 2 & 2 & -2 & -2 \\ 6 & 2 & 2 & -2 \end{bmatrix},$$

which, upon reordering the rows from last to first, becomes the Hankel matrix

$$\begin{bmatrix} 6 & 2 & 2 & -2 \\ 2 & 2 & -2 & -2 \\ 2 & -2 & -2 & -6 \\ -2 & -2 & -6 & -22 \end{bmatrix}.$$

In general, a Hankel matrix results from a Toeplitz matrix by reordering the rows from last to first. Specifically, the *Hankel matrix* $H(a_1, a_2, \ldots, a_{2n-1})$ is the matrix $H = [h_{ij}]$ of order n such that $h_{ij} = a_{i+j-1}$ for $i, j = 1, 2, \ldots, n$. Note that $H(a_1, a_2, \ldots, a_{2n-1})$ and $H(a_{2n-1}, \ldots, a_2, a_1)$ are related by a simultaneous permutation of rows and columns and thus have equal determinants. The Hankel matrix $H(c_{-(n-1)}, \ldots, c_{-1}, c_0, c_1, \ldots, c_{n-1})$ results from the Toeplitz matrix $T(c_{-(n-1)}, \ldots, c_{-1}, c_0, c_1, \ldots, c_{n-1})$ by reordering the rows from last to first, and thus their determinants differ only by a factor of $(-1)^{n(n-1)/2}$. Thus by Theorem 2.1,

$$\begin{aligned} \Pi_n &= \det(\widehat{B_n}) = (-1)^n \det\left(F_n + (F_n^{-1})^T\right) \\ &= (-1)^{n(n+1)/2} \det\left(H(r(n-2),\ldots,r(1),r(0)+1,-r(0)-1,-r(1),\ldots,-r(n-1))\right). \end{aligned} \tag{13}$$

We now turn to the evaluation of the determinant in (13). First we recall Dodgson's rule [5] for determinant calculation (see also [17]).[6] For a matrix A of order n, $A(i|j)$ denotes the matrix obtained from A by deleting row i and column j, and $A(i,j|k,l)$ denotes the matrix obtained from A by deleting rows i and j, and columns k and l.

[6] As alluded to in the introduction, Kuo [12] derived a formula for computing the number of perfect matchings in a planar bipartite graph which bears a strong resemblance to Dodgson's rule for determinants.

Lemma 4.4 *Let $A = [a_{ij}]$ be a matrix of order n. Then*

$$\det(A)\det(A(1,n|1,n)) = \det(A(1|1))\det(A(n|n)) - \det(A(1|n))\det(A(n|1)).$$

Applying Lemma 4.4 to a Hankel determinant we get the following identity.

Corollary 4.5

$$\det(H(a_1, a_2, \ldots, a_{2n-1}))\det(H(a_3, a_4, \ldots, a_{2n-3})) =$$

$$\det(H(a_3, a_4, \ldots, a_{2n-1}))\det(H(a_1, a_2, \ldots, a_{2n-3})) - \det(H(a_2, a_3, \ldots, a_{2n-2}))^2.$$

♣

In order to evaluate the determinant in (13) we shall need to evaluate a more general Hankel determinant of Schröder numbers $r(n)$. For $j, k \geq 1$, we define a matrix $H(j, k)$ of order $k + j$ by

$$H_{j,k} = H(r(2k-1), r(2k-2), \ldots, r(1), r(0)+1, -r(0)-1, -r(1), -r(2), \ldots, -r(2j-2)).$$

In addition, we define matrices $H_{0,k}$ of order k and $H_{j,0}$ of order j by

$$H_{0,k} = H(r(2k-1), \ldots, r(2), r(1)) \text{ and } H_{j,0} = H(-r(0)-1, -r(1), \ldots, -r(2j-2)).$$

To evaluate the determinants of the matrices $H_{k,j}$, we require the following result (Theorem 11 in [11] and Theorem 51.1 in [16]) which gives a method for computing Hankel determinants if one can find a certain continued fraction known as a *J-fraction.*

Lemma 4.6 *Let $(\mu_i; i \geq 0)$ be a sequence of numbers with generating function $\sum_{n=0}^{\infty} \mu_n x^n$ which can be expanded as a J-fraction:*

$$\sum_{n=0}^{\infty} \mu_n x^n = \cfrac{\mu_0}{1 + a_0 x - \cfrac{b_1 x^2}{1 + a_1 x - \cfrac{b_2 x^2}{1 + a_2 x - \cdots}}}.$$

Then

$$\det(H(\mu_0, \mu_1, \ldots, \mu_{2n-2})) = \mu_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-2}^2 b_{n-1}.$$

We first evaluate the determinants of $H_{0,k}$, and $H_{j,0}$.

Lemma 4.7 *For positive integers j and k,*

$$\det(H_{0,k}) = 2^{k(k+1)/2} \textit{ and } \det(H_{j,0}) = (-1)^j 2^{j(j+1)/2}.$$

Proof. As previously mentioned, the generating function for the Schröder numbers $(r(n); n \geq 0)$ is

$$\sum_{n=0}^{\infty} r(n)x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

Hence the generating function for the sequence of numbers $(r'(n) : n \geq 0)$, where $r'(0) = r(0) + 1 = 2$ and $r'(n) = r(n)$ for $n \geq 1$, is

$$f(x) = \sum_{n=0}^{\infty} r'(n)x^n = \frac{1 + x - \sqrt{1 - 6x + x^2}}{2x}.$$

Also the generating function for the sequence of numbers $(r(n+1) : n \geq 0)$ equals

$$g(x) = \sum_{n=0}^{\infty} r(n+1)x^n = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{2x^2}.$$

We have $\det(H_{0,k}) = \det(H(r(1), r(2), \ldots, r(2k-1))$ and $\det(H_{j,0}) = (-1)^j \det(H(r(0)+1, r(1), \ldots, r(2j-2))$. Thus we seek a J-fraction expansion of the generating functions $f(x)$ and $g(x)$. We first note that $w = g(x)$ is a solution of the equation $x^2w^2 - (1-3x)w + 2 = 0$ so that $w(1 - 3x - x^2w) = 2$. Therefore,

$$\begin{aligned}
w &= \frac{2}{1 - 3x - x^2 w} \\
w &= \frac{2}{1 - 3x - \frac{2x^2}{1-3x-x^2w}} \\
w &= \frac{2}{1 - 3x - \frac{2x^2}{1-3x-\frac{2x^2}{1-3x-x^2w}}} \\
&\cdots \\
w &= \frac{2}{1 - 3x - \frac{2x^2}{1-3x-\frac{2x^2}{1-3x-\frac{2x^2}{1-3x-\cdots}}}}
\end{aligned}$$

Thus in the J-fraction expansion as given in Lemma 4.6, we have $\mu_0 = b_1 = b_2 = b_3 = \cdots = 2$, and hence

$$\det(H(r(1), r(2), \ldots, r(2k-1)) = 2^{\sum_{i=1}^{k} i} = 2^{k(k+1)/2}.$$

Now we note that

$$f(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{2x}$$

$$
\begin{aligned}
&= \frac{8x}{2x(1+x+\sqrt{1-6x+x^2})} \\
&= \frac{2}{\frac{1+x}{2}+\frac{\sqrt{1-6x+x^2}}{2}} \\
&= \frac{2}{\frac{1+x}{2}-x^2w+\frac{1-3x}{2}} \\
&= \frac{2}{1-x-x^2w}.
\end{aligned}
$$

Inserting the J-fraction expansion of w, we obtain the J-fraction expansion of $f(x)$, and again $\mu_0 = 2 = b_1 = b_2 = b_3 = \cdots$. Hence

$$\det(H(r(0)+1, r(1), \ldots, r(2j-2)) = 2^{\sum_{i=1}^{j} i} = 2^{j(j+1)/2}.$$

This completes the proof of the lemma. ♣

We now evaluate the determinants of the matrices $H_{j,k}$.

Lemma 4.8 *For nonnegative integers k and j with $k+j \geq 1$, we have*

$$\det(H_{j,k}) = (-1)^j 2^{(k+j)(k+j+1)/2}. \qquad (14)$$

Proof. We prove (14) by induction on $l = j+k$. If $k=0$ or $j=0$, (14) follows from Lemma 4.7. We now assume that $k, j \geq 1$. If $k = j = 1$, then

$$\det H_{1,1} = \det \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} = -8 = (-1)^1 2^{2(2+1)/2}.$$

Now assume that $l \geq 3$. By Corollary 4.5 we have that $\det(H_{j,k})\det(H_{j-1,k-1})$ equals

$$\det(H_{j,k-1})\det(H_{j-1,k}) - \det(H(r(2k-2), \ldots, r(1), r(0)+1, -r(0)-1, \ldots, -r(2j-3))^2 =$$

$$\det(H_{j,k-1})\det(H_{j-1,k}) - \det(H_{k,j-1})^2,$$

the last since

$$\det(H(r(2k-2), \ldots, r(1), r(0)+1, -r(0)-1, \ldots, -r(2j-3))$$

$$= \det(H(-r(2j-3), \ldots, -r(0)-1, r(0)+1, \ldots, r(2k-2))).$$

Using the induction assumption, we now get

$$
\begin{aligned}
\det(H_{j,k})(-1)^{j-1}2^{(l-2)(l-1)/2} &= (-1)^j 2^{l(l-1)/2}(-1)^{j-1}2^{(l-1)l/2} - \left((-1)^k 2^{l(l-1)/2}\right)^2 \\
&= -2^{l(l-1)/2} - 2^{l(l-1)/2} = -2^{l(l-1)+1}.
\end{aligned}
$$

Therefore $\det(H_{j,k}) = (-1)^j 2^{l(l+1)/2}$ completing the induction. ♣

We now complete our proof that $\Pi_n = 2^{n(n+1)/2}$.

Theorem 4.9 *For $n \geq 1$,*

$$\det(\widehat{B_n}) = 2^{n(n+1)/2}.$$

Proof. By (13)

$$\det(\widehat{B_n}) = (-1)^{n(n+1)/2} \det\left(H(r(n-2), \ldots, r(1), r(0)+1, -r(0)-1, -r(1), \ldots, -r(n-1))\right).$$

For n an odd integer,

$$H(r(n-2), \ldots, r(1), r(0)+1, -r(0)-1, -r(1), \ldots, -r(n-1)) = H_{(n-1)/2,(n+1)/2}$$

so that by Lemma 4.8, its determinant equals

$$(-1)^{(n+1)/2} 2^{n(n+1)/2}.$$

For n an even integer,

$$\det\left(H(r(n-2), \ldots, r(1), r(0)+1, -r(0)-1, -r(1), \ldots, -r(n-1))\right)$$

equals

$$(-1)^n \det\left(H(r(n-1), \ldots, r(1), r(0)+1, -r(0)-1, r(1), \ldots, r(n-2))\right),$$

which equals the determinant of $H_{n/2,n/2}$. Hence

$$\det\left(H(r(n-2), \ldots, r(1), r(0)+1, -r(0)-1, -r(1), \ldots, -r(n-1))\right)$$

equals

$$(-1)^{n/2} 2^{n(n+1)/2}.$$

Therefore

$$\det(\widehat{B_n}) = (-1)^{n(n+1)/2} (-1)^{\lceil n/2 \rceil} 2^{n(n+1)/2} = 2^{n(n+1)/2}.$$

♣

Since Π_n equals $\det(\widehat{B_n})$, we immediately get our desired evaluation.

Corollary 4.10 *The number Π_n of tilings of the Aztec diamond of order n satisfies*

$$\Pi_n = 2^{n(n+1)/2}.$$

♣

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