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A Gale–Berlekamp permutation-switching problem

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ABSTRACT

In the spirit of the light switching game of Gale and Berlekamp, we define a light switching game based on permutations. We consider the game over the integers modulo k , that is, with light bulbs in an $n \times n$ formation, having k different intensities cyclically switching from 0 (off) to $(k - 1)$ (highest intensity) and then back to 0 (off). Under permutation switching, that is, adding a permutation matrix modulo k , given a particular initial pattern, we investigate both the smallest number $\mathcal{R}_{n,k}$ of on-lights (the covering radius of the code generated) and the smallest total intensity $\mathcal{I}_{n,k}$ that can be attained. We obtain an explicit formula for $\mathcal{I}_{n,k}$ when n is a multiple of k . We also determine $\mathcal{R}_{n,k}$ when k equals 2 and 3. In general, we obtain some bounds for $\mathcal{R}_{n,k}$ and $\mathcal{I}_{n,k}$.

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1. Introduction

The switching game of Gale and Berlekamp¹ is played on an $n \times n$ grid of light bulbs. The state (on or off) of each light bulb is individually controlled by a switch in the rear, but there are also $2n$ multiple-switches which change the state of each bulb in a row or each bulb in a column of the grid; call these *row switches* and *column switches*, respectively. Starting with an initial on/off pattern Θ of the n^2 light bulbs arranged by the rear switches, the game is to use the row and column switches to turn off all the lights or, if that is not possible, to obtain an on/off pattern with the smallest number of lights on.

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¹ Elwyn Berlekamp constructed a model of this game in the 1960s while at Bell Labs.

Table 1
Known values of \mathcal{R}_n .

n	1	2	3	4	5	6	7	8	9	10	11	12
\mathcal{R}_n	0	1	2	4	7	11	16	22	27	35	43	54

Let the *weight* of a light pattern Θ equal the number $w(\Theta)$ of on-lights in Θ , and let $w^*(\Theta)$ equal the smallest number of on-lights achievable by row and column switches starting with the pattern Θ . The *extremal problem* is to determine \mathcal{R}_n where

$$\mathcal{R}_n = \max\{w^*(\Theta) : \Theta \text{ an } n \times n \text{ light pattern}\}.$$

Thus every light pattern can be reduced by row and column switches to a pattern with at most \mathcal{R}_n on-lights, and \mathcal{R}_n is the smallest integer with this property.

According to [2], the values of \mathcal{R}_n known exactly are given in Table 1.

As noted in [3,7], \mathcal{R}_n is the *covering radius* of the binary code determined by the row and column switches. This code, the *Gale–Berlekamp code*, consists of all $n \times n$ matrices over the binary field \mathbb{Z}_2 generated by the $2n \times n$ (0, 1)-matrices with exactly n 1’s where these 1’s are either in a row or in a column. The covering radius of the code is by definition the smallest integer r such that every $n \times n$ matrix over \mathbb{Z}_2 differs from some codeword in at most r positions.

The Gale–Berlekamp code has dimension equal to $2n - 1$. For example, if $n = 3$, it is generated by

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and their transposes, where the binary sum of the three displayed matrices equals the binary sum of their transposes, and thus one is redundant. Additional references include [8] and [4] where a hardness result for decoding the Gale–Berlekamp code is obtained.

Schauz in [5,6] investigates a generalization of the Gale–Berlekamp switching game to higher dimensions, namely a $q \times q \times \dots \times q$ grid of lights, where q is a prime power, and where the lights have k different states. In this paper we introduce a switching game of the Gale–Berlekamp type, but with different, and many more in number, multiple-switches.

Consider an $n \times n$ grid of light bulbs each having k different states $0, 1, \dots, k - 1$, ranging from 0 (off) to $(k - 1)$ (brightest). Each light bulb is controlled by an individual switch which, as in physical multi-way bulbs, moves it cyclically one click at a time, from off to brightest and then back to off again. Let $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ be the ring of integers modulo k . We then consider the states of each bulb as the elements of \mathbb{Z}_k . A *grid-state* is any setting of the states of all of the light bulbs of a grid achieved by the individual switches. Thus a grid-state can be viewed as an $n \times n$ matrix $\Theta = [\theta_{ij}]$ over \mathbb{Z}_k , where θ_{ij} is the state of the light bulb in position (i, j) of the grid. We consider two parameters associated with such a grid-state Θ :

- (i) The *weight* $w(\Theta) = |\{(i, j) : \theta_{ij} \neq 0 \text{ where } 1 \leq i, j \leq n\}|$ equal to the number of light bulbs that are on (any state from 1 to $k - 1$).
- (ii) The *intensity* $\iota(\Theta)$ equal to $\sum_{i,j=1}^n \theta_{ij}$.

Here and in what follows, in order to avoid any confusion, we distinguish arithmetic in \mathbb{Z}_k from real arithmetic by using “ $\equiv \text{mod } k$ ” or saying “computed in \mathbb{Z}_k ” when arithmetic is done in the integers modulo k . Sometimes it is convenient to denote addition modulo k by $+_k$. Thus a lack of such identifiers implies that the arithmetic is real arithmetic. In particular, in (ii) above, the intensity is computed in real arithmetic. (As one of the referees suggested, there are other candidates for parameters that could be considered; these include the intensity as computed using arithmetic in \mathbb{Z}_k , or a complex intensity by regarding \mathbb{Z}_k as the group of k th roots of unity in the plane.)

In our switching game, there are $n!$ multiple-switches and these correspond to the $n \times n$ permutation matrices. Thus for each permutation $\sigma = (i_1, i_2, \dots, i_n)$ of $\{1, 2, \dots, n\}$, there is a multiple-switch S_σ which adds 1 modulo k to the state of the light bulbs in positions $(1, i_1), (2, i_2), \dots, (n, i_n)$ of the grid. Thus a click of a multiple switch acts like a click on the individual switch of each light

bulb it affects. In terms of the defined parameters, given an initial grid-state $\Theta = [\theta_{ij}]$, there are two quantities to be considered:

- (i') $w^*(\Theta)$, the *minimum weight achievable by permutation switches* starting with the pattern Θ .
- (ii') $l^*(\Theta)$, the *minimum intensity achievable by permutation switches* starting with the pattern Θ .

(In case $k = 2$, the binary case, the two quantities in (i) and (ii) are equal, as are those in (i') and (ii').) These quantities give rise to two *extremal problems*:

- I. Determine $\mathcal{R}_{n,k} = \max\{w^*(\Theta) : \Theta \text{ an } n \times n \text{ grid-state}\}$.
- II. Determine $\mathcal{I}_{n,k} = \max\{l^*(\Theta) : \Theta \text{ an } n \times n \text{ grid-state}\}$.

There exists a grid-state with $\mathcal{R}_{n,k}$ lights on (in one of the states $1, 2, \dots, k - 1$), so that no sequence of permutation switches can reduce the number of on-lights. Likewise, there exists a grid state with intensity $\mathcal{I}_{n,k}$ such that no sequence of permutation switches can reduce the intensity. The numbers $\mathcal{R}_{n,k}$ and $\mathcal{I}_{n,k}$ are minimum for these respective properties.

Let $M_n(\mathbb{Z}_k)$ denote the set of $n \times n$ matrices over \mathbb{Z}_k , that is, the set of possible $n \times n$ grid-states. Using matrix addition, $M_n(\mathbb{Z}_k)$ is a finite abelian group, a direct sum of n^2 copies of \mathbb{Z}_k . The $n \times n$ permutation matrices generate a subgroup $P_n(\mathbb{Z}_k)$, which we can be regarded as a k -ary code. Recall that in coding theory, the *weight* $\text{wt}(x)$ of an m -vector $x \in \mathbb{Z}_k^m$ equals the number of nonzero coordinates of x , and the *distance* $\text{dist}(x, y)$ between two m -vectors x and y is the number of coordinates in which x and y differ (the weight of the difference vector $x - y$). The distance of a vector x to a code C is the smallest distance of x to a codeword,

$$\min\{\text{dist}(x, c) : c \in C\},$$

and this equals the minimum weight of a vector in the coset $x + C$ containing x . The *covering radius* of a code C is the largest distance of a vector in \mathbb{Z}_k^m to the code:

$$\max\{\min\{\text{dist}(x, c) : c \in C\} : x \in \mathbb{Z}_k^m\}.$$

In this language, $w^*(\Theta)$ is the *distance* of Θ to the code $P_n(\mathbb{Z}_k)$, thus the minimum weight of a matrix in the coset $\Theta + P_n(\mathbb{Z}_k)$. $\mathcal{R}_{n,k}$ equals the *covering radius* of $P_n(\mathbb{Z}_k)$, thus the maximum of the minimum weights of its cosets. The quantities $l^*(\Theta)$ and $\mathcal{I}_{n,k}$ are usually not considered in coding theory. In our context, $l^*(\Theta)$ is the minimum intensity of a matrix in the coset $\Theta + P_n(\mathbb{Z}_k)$, and $\mathcal{I}_{n,k}$ is the maximum of the minimum intensities of the cosets. Thus every light pattern Θ can be reduced by permutation switches to a light pattern with intensity at most equal to $\mathcal{I}_{n,k}$ and $\mathcal{I}_{n,k}$ is the smallest integer with this property.

Our goal in this paper is to determine $\mathcal{R}_{n,k}$ and $\mathcal{I}_{n,k}$, of which we only partially succeed. It turns out, perhaps surprisingly, that computing $\mathcal{R}_{n,k}$ is more difficult than computing $\mathcal{I}_{n,k}$. This may be due to the fact that in the case of $\mathcal{I}_{n,k}$, algebra, and not combinatorics, plays a more prominent role. In the next section, we develop some preliminaries needed to carry out our investigations.

2. Preliminaries

The possible grid-states have been identified with the k^{n^2} matrices $\Theta = [\theta_{ij}]$ of the abelian group $M_n(\mathbb{Z}_k)$. Each Θ has a *row sum vector* $R = (r_1, r_2, \dots, r_n)$ (r_i is the sum, computed in \mathbb{Z}_k , of the entries in row i) and *column sum vector* $S = (s_1, s_2, \dots, s_n)$ defined similarly.

Sequences of permutation switches can be identified with the elements $A = [a_{ij}]$ of the subgroup $P_n(\mathbb{Z}_k)$ of $M_n(\mathbb{Z}_k)$ generated by the $n \times n$ permutation matrices. Thus a sequence of permutation switches applied to Θ replaces Θ with $\Theta + A$:

$$\Theta \rightarrow \Theta + A \pmod{k},$$

where A is the mod k sum of permutation matrices.

We first consider the subgroup $P_n(\mathbb{Z}_k)$ and prove the following lemma.

Lemma 2.1. *Let $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ be two n -vectors in \mathbb{Z}_k^n . Then there exists a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector $R \in \mathbb{Z}_k^n$ and column sum vector $S \in \mathbb{Z}_k^n$ if and only if*

$$r_1 + r_2 + \dots + r_n \equiv s_1 + s_2 + \dots + s_n \pmod{k}. \tag{1}$$

Moreover, when (1) holds, there exists a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector R and column sum vector S having at most $2n - 1$ nonzero entries.

Proof. This lemma is well-known if we use real arithmetic rather than “mod k ” arithmetic (see e.g. [1] and also the proof of Lemma 2.6 below for more details). In addition, there is a well-known, simple recursive algorithm to construct a matrix with at most $2n - 1$ nonzero integral entries whose row and column sum vectors in real arithmetic are R and S , respectively.

The condition (1) is certainly necessary for existence of a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector R and column sum vector S , since each side is the mod k sum of the entries of such a matrix. Now suppose this condition holds. Then

$$r_1 + r_2 + \dots + r_n = s_1 + s_2 + \dots + s_n + pk$$

for some integer p . We assume that $p \geq 0$; otherwise, we may interchange R and S . Thus there exists an $n \times n$ matrix A over \mathbb{Z} with row sum vector R and column sum vector $S' = (s_1, s_2, \dots, s_{n-1}, s_n + pk)$ with at most $2n - 1$ nonzero entries. Taking the entries of A mod k , we obtain the desired matrix with at most $2n - 1$ nonzero entries in \mathbb{Z}_k . \square

Lemma 2.1 allows us to characterize the matrices in $P_n(\mathbb{Z}_k)$.

Lemma 2.2. An $n \times n$ matrix $A \in M_n(\mathbb{Z}_k)$ belongs to $P_n(\mathbb{Z}_k)$ if and only if all its row and column sums computed in \mathbb{Z}_k equal the same number.

Proof. A matrix in $P_n(\mathbb{Z}_k)$ which is the sum of t permutation matrices has all its row and column sums equal to t mod k . Now let A be a matrix in $M_n(\mathbb{Z}_k)$ with all row and column sums equal to c mod k . Considering A as an $n \times n$ matrix in \mathbb{Z} , we see that its row and column sum vectors are of the form

$$R' = (c + r_1k, c + r_2k, \dots, c + r_nk) \quad \text{and} \quad S' = (c + s_1k, c + s_2k, \dots, c + s_nk)$$

where $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ are nonnegative integral vectors. We have $\sum_{i=1}^n (c + r_ik) = \sum_{i=1}^n (c + s_ik)$, and thus

$$\sum_{i=1}^n r_i = \sum_{i=1}^n s_i. \tag{2}$$

Let p be an integer with $p \geq \max\{r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n\}$, and consider the integral vectors

$$R'' = (p - r_1, p - r_2, \dots, p - r_n) \quad \text{and} \quad S'' = (p - s_1, p - s_2, \dots, p - s_n).$$

It follows from (2) and Lemma 2.1 that there is a nonnegative integral matrix B with row sum vector R'' and column sum vector S'' . The nonnegative integral matrix $A + kB$ has all row and column sums equal to $c + pk$ and hence is the real sum of $c + pk$ permutation matrices. Since A is equal to $A + kB$ mod k , it follows that A is the “mod k ” sum of permutation matrices and hence belongs to $P_n(\mathbb{Z}_k)$. \square

Corollary 2.3. Let $\Theta_1, \Theta_2 \in M_n(\mathbb{Z}_k)$ where the row sum vectors, respectively, column sum vectors, of Θ_1 and Θ_2 are equal mod k . Then Θ_1 and Θ_2 belong to the same coset of $M_n(\mathbb{Z}_k)$ with respect to the subgroup $P_n(\mathbb{Z}_k)$.

Proof. We have that $\Theta_1 - \Theta_2$ is in $M_n(\mathbb{Z}_k)$ with all row and column sums equal to 0 mod k . By Lemma 2.2, $\Theta_1 - \Theta_2 \in P_n(\mathbb{Z}_k)$ and thus Θ_1 and Θ_2 belong to the same coset. \square

Corollary 2.4. For all n and k , we have

$$\mathcal{R}_{n,k} \leq 2n - 1.$$

Proof. Applying Lemma 2.1 and Corollary 2.3, we conclude that each coset contains a matrix with at most $2n - 1$ nonzero entries. Thus $\mathcal{R}_{n,k} \leq 2n - 1$. \square

Our problem is to determine the maximum weight and maximum intensity of a coset of $M_n(\mathbb{Z}_k)$ with respect to the subgroup $P_n(\mathbb{Z}_k)$ generated by the $n \times n$ permutation matrices and characterized in Lemma 2.2; recall the weight and intensity of a coset are, respectively, the minimum weight and

minimum intensity of the grid-states in the coset. The cosets of $M_n(\mathbb{Z}_k)$ with respect to $P_n(\mathbb{Z}_k)$ are partially described in [Corollary 2.3](#). We now completely identify the elements in these cosets.

For $r = 0, 1, \dots, k - 1$, let $P_n^r(\mathbb{Z}_k)$ be the set of matrices in $P_n(\mathbb{Z}_k)$ all of whose row and column sums equal $r \pmod k$. Using [Lemma 2.2](#) we have the pairwise disjoint union

$$P_n(\mathbb{Z}_k) = P_n^0(\mathbb{Z}_k) \cup P_n^1(\mathbb{Z}_k) \cup \dots \cup P_n^{k-1}(\mathbb{Z}_k).$$

The sets $P_n^0(\mathbb{Z}_k), P_n^1(\mathbb{Z}_k), \dots, P_n^{k-1}(\mathbb{Z}_k)$ are the cosets of $P_n(\mathbb{Z}_k)$ with respect to its subgroup $P_n^0(\mathbb{Z}_k)$; in fact,

$$P_n^r(\mathbb{Z}_k) = rI_n + P_n^0(\mathbb{Z}_k), \quad (r = 0, 1, \dots, k - 1),$$

where I_n is the $n \times n$ identity matrix. We thus have the following corollary which describes the cosets of $M_n(\mathbb{Z}_k)$ relative to its subgroup $P_n(\mathbb{Z}_k)$.

Corollary 2.5. *Let A be a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector $R = (r_1, r_2, \dots, r_n)$ and column sum vector $S = (s_1, s_2, \dots, s_n) \pmod k$, where therefore $r_1 + r_2 + \dots + r_n \equiv s_1 + s_2 + \dots + s_n \pmod k$. The coset of $P_n(\mathbb{Z}_k)$ containing A consists of all matrices in $M_n(\mathbb{Z}_k)$ whose row sum and column sum vector pairs $\pmod k$ are one of*

$$(R + r(1, 1, \dots, 1), S + r(1, 1, \dots, 1)) \quad \text{for } r = 0, 1, \dots, k - 1. \tag{3}$$

The number of matrices in each $P_n^r(\mathbb{Z}_k)$ equals $k^{(n-1)^2}$, since the entries in the leading $(n - 1) \times (n - 1)$ submatrix can be arbitrarily prescribed and then the remaining $(2n - 1)$ entries are uniquely determined by the condition that all row and column sums equal $r \pmod k$. The number of matrices in $P_n(\mathbb{Z}_k)$, and thus in each of its cosets in $M_n(\mathbb{Z}_k)$ equals $k \cdot k^{(n-1)^2} = k^{(n-1)^2+1}$.

Those matrices with row sum vector R and column sum vector S (the case $r = 0$ in (3)) form a coset of $M_n(\mathbb{Z}_k)$ with respect to its subgroup $P_n^0(\mathbb{Z}_k)$. The intensity of a coset with respect to $P_n^0(\mathbb{Z}_k)$ is easily determined. First we prove the following lemma for the set $M_{m,n}(\mathbb{Z}_k)$ of $m \times n$ matrices over \mathbb{Z}_k .

Lemma 2.6. *Let $R = (r_1, r_2, \dots, r_m)$ and $S = (s_1, s_2, \dots, s_n)$ be vectors in \mathbb{Z}_k^n with $r_1 + r_2 + \dots + r_m \equiv s_1 + s_2 + \dots + s_n \pmod k$. Then there is a matrix in $M_{m,n}(\mathbb{Z}_k)$ with row sum vector R and column sum vector S the sum of whose entries in real arithmetic equals*

$$\max\{r_1 + r_2 + \dots + r_m, s_1 + s_2 + \dots + s_n\}. \tag{4}$$

Proof. We use the constructive algorithm mentioned in the proof of [Lemma 2.1](#). Assume without loss of generality that the integer r_1 is at most the integer s_1 , so that $s_1 - r_1$ is one of $0, 1, \dots, k - 1$. The proof is by induction on $m + n \geq 2$. If $m + n = 2$ the result is trivial, so without loss of generality we assume that $m \geq 2$. We construct the desired matrix A in $M_n(\mathbb{Z}_k)$ with row sum vector R and column sum vector S by setting the first row equal to $r_1, 0, \dots, 0$, thereby requiring us to construct the matrix $A' \in M_{m-1,n}(\mathbb{Z}_k)$ in (5) below with row sum vector $R' = (r_2, \dots, r_m) \in \mathbb{Z}_k^{m-1}$ and column sum vector $S' = (s_1 - r_1, s_2, \dots, s_n) \in \mathbb{Z}_k^n$:

$$\left[\begin{array}{cccccc} r_1 & 0 & 0 & \dots & 0 \\ \hline & & A' & & \end{array} \right]. \tag{5}$$

Since $r_2 + \dots + r_m \equiv (s_1 - r_1) + s_2 + \dots + s_n \pmod k$, it follows by induction that such a matrix A' exists with the sum of its entries in real arithmetic equal to

$$\max\{r_2 + \dots + r_m, (s_1 - r_1) + s_2 + \dots + s_n\}.$$

Thus there exists a matrix in $M_{m,n}(\mathbb{Z}_k)$ with row sum vector R and column sum vector S , the real sum of whose entries equals

$$\begin{aligned} r_1 + \max\{r_2 + \dots + r_m, (s_1 - r_1) + s_2 + \dots + s_n\} \\ = \max\{r_1 + r_2 + \dots + r_m, s_1 + s_2 + \dots + s_n\}. \end{aligned}$$

Hence the lemma follows by induction. \square

The following corollary determines the intensity of a coset of $M_n(\mathbb{Z}_k)$ with respect to its subgroup $P_n^0(\mathbb{Z}_k)$.

Corollary 2.7. *Let $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ be vectors in \mathbb{Z}_k^n with $r_1 + r_2 + \dots + r_n \equiv s_1 + s_2 + \dots + s_n \pmod k$. Then the minimum intensity of a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector R and column sum vector S equals*

$$\max\{r_1 + r_2 + \dots + r_n, s_1 + s_2 + \dots + s_n\}. \tag{6}$$

The intensity of the coset containing a matrix with row sum vector R and column sum vector S equals

$$\min \left\{ \max \left\{ \sum_{i=1}^n (r_i +_k r), \sum_{i=1}^n (s_i +_k r) \right\} : r = 0, 1, \dots, k \right\}. \tag{7}$$

Proof. The integer in (6) is certainly a lower bound on the intensity of a matrix in $M_n(\mathbb{Z}_k)$ with row sum vector R and column sum vector $S \pmod k$. By Lemma 2.6 there exists a matrix with row sum vector R and column sum vector S with intensity (6). (7) is an immediate consequence of (6) and Corollary 2.5. \square

We can now obtain an expression for the maximum intensity of a grid-state that cannot be reduced by permutation switching.

Theorem 2.8. *For all n and k , we have*

$$\mathcal{I}_{n,k} = \max \left\{ \min \left\{ \max \left\{ \sum_{i=1}^n (r_i +_k r), \sum_{i=1}^n (s_i +_k r) \right\} : r = 0, 1, \dots, k - 1 \right\} : R, S \right\}, \tag{8}$$

where the outer maximum is taken over all $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ in \mathbb{Z}_k^n such that $r_1 + r_2 + \dots + r_n \equiv s_1 + s_2 + \dots + s_n \pmod k$.

Proof. The theorem is an immediate consequence of (7) of Corollary 2.7 and the definition of $\mathcal{I}_{n,k}$ (the outer maximum). \square

3. Computation of $\mathcal{I}_{n,k}$

Given n and k , Theorem 2.8 gives an expression for the maximum intensity $\mathcal{I}_{n,k}$ that cannot be reduced by permutation switching. We now consider the computation of $\mathcal{I}_{n,k}$. First we consider the simpler binary case (so $k = 2$ and a light is either on or off) where we know that $\mathcal{R}_{n,2} = \mathcal{I}_{n,2}$.

Theorem 3.1. *For all $n \geq 1$, $\mathcal{R}_{n,2} = \mathcal{I}_{n,2} = \begin{cases} n & \text{if } n \text{ is even,} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$*

Proof. The row sum vector R and column sum vector S of a matrix Θ in $M_n(\mathbb{Z}_2)$ consist of 0s and 1s where the number p of 1s in R and the number q of 1s in S have the same parity. It follows from Corollaries 2.3 and 2.7 that $w^*(\Theta) = \max\{p, q\} \leq n$. Thus $\mathcal{R}_{n,2} \leq n$.

First suppose that n is even, and let

$$\Theta = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

where $R = (1, 1, \dots, 1)$, $S = (0, 0, \dots, 0)$ and $w(\Theta) = n$. Since a permutation matrix has row and column sum vectors equal to $(1, 1, \dots, 1)$, it follows that each matrix in the coset containing Θ has this row sum vector and this column sum vector mod 2, or the row sum vector and column sum vector are interchanged. Hence $w^*(\Theta) \geq n$ and so $w^*(\Theta) = n$. Therefore $\mathcal{R}_{n,2} = n$.

Now suppose that n is odd. If R and S both equal $(1, 1, \dots, 1)$, then $I_n + \Theta$ is in the same coset as Θ and, since it has all row and column sums equal to 0 mod 2, $I_n + \Theta$ is in $P_n(\mathbb{Z}_2)$. Thus $\Theta \in P_n(\mathbb{Z}_2)$ and

hence $w^*(\Theta) = 0$. Otherwise, at least one of R and S contains a 0, say R contains a 0. If S also contains a 0, then by Corollary 2.7, $w(\Theta) \leq n - 1$. Suppose that $S = (1, 1, \dots, 1)$. Then since n is odd, R also contains a 1, and hence both the row and column sum vectors of $I_n + \Theta$ contain at least one 0 mod 2. By Corollary 2.7 again, $w^*(\Theta) \leq n - 1$ and so $\mathcal{R}_{n,2} \leq n - 1$. The matrix

$$\Theta = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

has row sum vector $(0, 1, \dots, 1)$ and column sum vector $(0, 0, \dots, 0)$ mod 2, and each matrix in its coset has these as row and column sum vectors mod 2 or has row sum vector $(1, 0, \dots, 0)$ and column sum vector $(1, 1, \dots, 1)$ mod 2. Thus by Corollary 2.7, $w^*(\Theta) = n - 1$ and so $\mathcal{R}_{n,2} = n - 1$. \square

To determine the maximum intensity $\mathcal{I}_{n,k}$ in general, we proceed as follows. Let $D = (d_1, d_2, \dots, d_n)$ be an arbitrary vector in \mathbb{Z}_k^n . For $i = 1, 2, \dots, k$ we define D_i recursively by: $D_1 = D$ and for $2 \leq i \leq k$, D_i is the vector obtained from D_{i-1} by adding the vector $(1, 1, \dots, 1)$ mod k . Let u_i equal the real sum of the components of D_i ($i = 1, 2, \dots, k$), and let $U = (u_1, u_2, \dots, u_k)$. We calculate that

$$\sum_{i=1}^k u_i = n \binom{k}{2}, \tag{9}$$

since in each of the n coordinates, each of the elements $0, 1, \dots, k-1$ of \mathbb{Z}_k occurs once in some order. Let \widehat{D} be the $k \times n$ matrix whose rows from first to last are D_1, D_2, \dots, D_k . Similarly, given a vector $E \in \mathbb{Z}_k^n$, we define a $k \times n$ matrix \widehat{E} with rows $E_1 = E, E_2, \dots, E_k$, and $V = (v_1, v_2, \dots, v_k)$ where v_i is equal to the real sum of the components of E_i ($i = 1, 2, \dots, k$).

We first consider the case $n = k$ so that in (9) the sum is $k \binom{k}{2}$. Then \widehat{D} and \widehat{E} are in $M_k(\mathbb{Z}_k)$. By Corollary 2.5, provided $u_1 + u_2 + \dots + u_k \equiv v_1 + v_2 + \dots + v_k \pmod k$, the pairs (D_i, E_i) are the mod k row sum and column sum vectors of a coset of $M_k(\mathbb{Z}_k)$ with respect to $P_k(\mathbb{Z}_k)$. We start with an example.

Example 3.2. Let $n = k = 8$ and let $D = (4, 4, 4, 4, 4, 5, 6, 7)$. Then

$$[\widehat{D} \parallel U^t] = \left[\begin{array}{c|c} D_1 & u_1 \\ D_2 & u_2 \\ D_3 & u_3 \\ D_4 & u_4 \\ D_5 & u_5 \\ D_6 & u_6 \\ D_7 & u_7 \\ D_8 & u_8 \end{array} \right] = \left[\begin{array}{cccccccc|c} 4 & 4 & 4 & 4 & 4 & 5 & 6 & 7 & 38 \\ 5 & 5 & 5 & 5 & 5 & 6 & 7 & 0 & 38 \\ 6 & 6 & 6 & 6 & 6 & 7 & 0 & 1 & 38 \\ 7 & 7 & 7 & 7 & 7 & 0 & 1 & 2 & 38 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 14 \\ 2 & 2 & 2 & 2 & 2 & 3 & 4 & 5 & 22 \\ 3 & 3 & 3 & 3 & 3 & 4 & 5 & 6 & 30 \end{array} \right].$$

Let $E = (0, 0, 0, 0, 0, 1, 2, 3) = D_5$, and so \widehat{E} is the 8×8 matrix

$$[\widehat{E} \parallel V^t] = \left[\begin{array}{c|c} E_1 & v_1 \\ E_2 & v_2 \\ E_3 & v_3 \\ E_4 & v_4 \\ E_5 & v_5 \\ E_6 & v_6 \\ E_7 & v_7 \\ E_8 & v_8 \end{array} \right] = \left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 14 \\ 2 & 2 & 2 & 2 & 2 & 3 & 4 & 5 & 22 \\ 3 & 3 & 3 & 3 & 3 & 4 & 5 & 6 & 30 \\ 4 & 4 & 4 & 4 & 4 & 5 & 6 & 7 & 38 \\ 5 & 5 & 5 & 5 & 5 & 6 & 7 & 0 & 38 \\ 6 & 6 & 6 & 6 & 6 & 7 & 0 & 1 & 38 \\ 7 & 7 & 7 & 7 & 7 & 0 & 1 & 2 & 38 \end{array} \right].$$

We consider the rows of \widehat{D} and those of \widehat{E} as the potential row and column sum vectors, respectively, of a coset of matrices in $M_8(\mathbb{Z}_8)$ with respect to $P_8(\mathbb{Z}_8)$. The rows of \widehat{E} are a (cyclic) permutation of the rows of \widehat{D} , and it follows that $u_i \equiv v_i \pmod 8$ for $i = 1, 2, \dots, 8$. There is a matrix $\Theta_8 \in M_8(\mathbb{Z}_8)$

with row sum vector D_1 and column sum vector E_1 whose coset with respect to $P_8(\mathbb{Z}_8)$ consists of all matrices in $M_8(\mathbb{Z}_8)$ with row and column sum pairs mod k equal to (D_i, E_i) for some $i \in \{1, 2, \dots, 8\}$. We calculate that by (7) the minimum intensity of a matrix in the coset containing Θ_8 , the minimum intensity attainable by applying permutation switches to Θ , is the minimum of the eight numbers

$$\begin{aligned} &\max\{38, 6\}, \max\{38, 14\}, \max\{38, 22\}, \max\{38, 30\}, \\ &\max\{6, 38\}, \max\{14, 38\}, \max\{22, 38\}, \max\{30, 38\}, \end{aligned}$$

and thus is 38.

Similarly, in the odd case with $n = k = 7$, $D = (3, 3, 3, 3, 4, 5, 6)$, and $E = (0, 0, 0, 0, 1, 2, 3)$, we get

$$\left[\widehat{D} \parallel U^t \right] = \left[\begin{array}{c|c} D_1 & u_1 \\ D_2 & u_2 \\ D_3 & u_3 \\ D_4 & u_4 \\ D_5 & u_5 \\ D_6 & u_6 \\ D_7 & u_7 \end{array} \right] = \left[\begin{array}{cccccccc|c} 3 & 3 & 3 & 3 & 4 & 5 & 6 & 27 \\ 4 & 4 & 4 & 4 & 5 & 6 & 0 & 27 \\ 5 & 5 & 5 & 5 & 6 & 0 & 1 & 27 \\ 6 & 6 & 6 & 6 & 0 & 1 & 2 & 27 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 & 13 \\ 2 & 2 & 2 & 2 & 3 & 4 & 5 & 20 \end{array} \right]$$

and

$$\left[\widehat{E} \parallel V^t \right] = \left[\begin{array}{c|c} E_1 & v_1 \\ E_2 & v_2 \\ E_3 & v_3 \\ E_4 & v_4 \\ E_5 & v_5 \\ E_6 & v_6 \\ E_7 & v_7 \end{array} \right] = \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 2 & 3 & 4 & 13 \\ 2 & 2 & 2 & 2 & 3 & 4 & 5 & 20 \\ 3 & 3 & 3 & 3 & 4 & 5 & 6 & 27 \\ 4 & 4 & 4 & 4 & 5 & 6 & 0 & 27 \\ 5 & 5 & 5 & 5 & 6 & 0 & 1 & 27 \\ 6 & 6 & 6 & 6 & 0 & 1 & 2 & 27 \end{array} \right].$$

Thus the minimum intensity of a matrix in the coset containing a matrix Θ_7 with row sum vector D and column sum vector V is the minimum of the seven numbers

$$\begin{aligned} &\max\{27, 6\}, \max\{27, 13\}, \max\{27, 20\}, \max\{27, 27\}, \\ &\max\{6, 27\}, \max\{13, 27\}, \max\{20, 27\} \end{aligned}$$

and thus is 27. \square

With these examples in mind, we now determine $J_{k,k}$, but first we prove a lemma.

Let a_1, a_2, \dots, a_n be n numbers ordered so that $a_1 \leq a_2 \leq \dots \leq a_n$. Then their median M is the $\lceil (n + 1)/2 \rceil$ largest number $a_{\lceil (n+1)/2 \rceil}$. Thus M is the middle number if n is odd and the $(n/2 + 1)$ st number if n is even. There are at least as many numbers equal to or greater than the median as there are numbers less than the median. In Example 3.2, the median of the first U and V ($n = 8$) is 38, and the median of the second U and V ($n = 7$) is 27.

Lemma 3.3. Let c_1, c_2, \dots, c_n and d_1, d_2, \dots, d_n be two sequences of real numbers each of whose median is at most α . Then

$$\min \{ \max\{c_1, d_1\}, \max\{c_2, d_2\}, \dots, \max\{c_n, d_n\} \} \leq \alpha.$$

Proof. Suppose to the contrary that $\max\{c_i, d_i\} > \alpha$ for all $i = 1, 2, \dots, n$. First assume that n is even. Then there is a partition of $\{1, 2, \dots, n\}$ into two sets K and L such that

$$c_i > \alpha \ (i \in K) \quad \text{and} \quad d_i > \alpha \ (i \in L).$$

Either $|K| \geq n/2$ or $|L| \geq n/2$. If $|K| \geq n/2$, then the median of c_1, c_2, \dots, c_n is strictly greater than α , a contradiction. A similar contradiction results if $|L| \geq n/2$.

Now suppose that n is odd. Then we have the same situation as above except that either $|K| \geq (n + 1)/2$ or $|L| \geq (n + 1)/2$, and the median of c_1, c_2, \dots, c_n or the median of d_1, d_2, \dots, d_n is strictly greater than α , a contradiction. \square

Theorem 3.4. Let $k \geq 2$ be an integer. Then

$$J_{k,k} = \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}. \tag{10}$$

Proof. Consider sequences $D = (d_1, d_2, \dots, d_k)$ and $E = (e_1, e_2, \dots, e_k)$ in \mathbb{Z}_k^k with $d_1 + d_2 + \dots + d_k \equiv e_1 + e_2 + \dots + e_k \pmod k$. By Lemma 2.1 and as in Example 3.2, D and E are row and column sum vectors of a matrix in $M_k(\mathbb{Z}_k)$. As in that example let $D_1 = D$ and let D_i be obtained from D_{i-1} by adding $(1, 1, \dots, 1) \pmod k$ to D_{i-1} ($i = 2, 3, \dots, k$). Note that D_1 is obtained from D_k by adding $(1, 1, \dots, 1) \pmod k$. Thus we regard D_1, D_2, \dots, D_k as a cyclic sequence with D_1 following D_k . Define E_1, E_2, \dots, E_k analogously. For $i = 1, 2, \dots, k$, let u_i be the real sum of the entries of D_i with α equal to the median of u_1, u_2, \dots, u_k , and let v_i equal the real sum of the entries of E_i with α' equal to the median of v_1, v_2, \dots, v_k .

Since D_i is obtained from D_{i-1} by adding $(1, 1, \dots, 1) \pmod k$, we have $u_{i-1} \geq u_i - k$ (indices taken mod k also) with equality if and only if no entry of D_{i-1} equals $k - 1$. Let j be any index such that $u_j = \alpha$. Then

$$u_{j-1} \geq \alpha - k, u_{j-2} \geq \alpha - 2k, \dots, u_{j-\lfloor \frac{k}{2} \rfloor} \geq \alpha - \lfloor \frac{k}{2} \rfloor k. \tag{11}$$

These inequalities (11), along with (9) and our definition of median, imply that

$$\begin{aligned} k \binom{k}{2} &= \sum_{i=1}^k u_i \\ &\geq \lfloor \frac{k+1}{2} \rfloor \alpha + \left(\lfloor \frac{k}{2} \rfloor \alpha - k \left(1 + 2 + \dots + \lfloor \frac{k}{2} \rfloor \right) \right) \\ &= k\alpha - k \binom{\lceil \frac{k+1}{2} \rceil}{2}. \end{aligned} \tag{12}$$

Equality holds in (12) if and only if all the inequalities in (11) are equalities. Thus

$$\alpha \leq \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}; \tag{13}$$

equality holds in (13) if and only if, after possibly changing the start of the cyclic order of u_1, u_2, \dots, u_k ,

$$(u_1, u_2, \dots, u_k) = \left(\underbrace{\alpha, \dots, \alpha}_{\lfloor \frac{k+1}{2} \rfloor}, \alpha - k, \alpha - 2k, \dots, \alpha - \lfloor \frac{k}{2} \rfloor k \right), \tag{14}$$

$$\alpha = \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}.$$

Likewise, the median α' of v_1, v_2, \dots, v_k satisfies

$$\alpha' \leq \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}, \tag{15}$$

with similar conclusions. It now follows from (13) and (15) and the fact that α and α' are the medians of the components of U and V , that for each $\Theta \in M_k(\mathbb{Z}_k)$ with row sum vector U and column sum vector V , we have

$$i^*(\Theta) = \min \{ \max \{ (u_i, v_i) \} \} \leq \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}. \tag{16}$$

If equality holds, then after starting the cyclic order of u_1, u_2, \dots, u_k and v_1, v_2, \dots, v_k with the same index, (u_1, u_2, \dots, u_k) is given as in (14) and

$$(v_1, v_2, \dots, v_k) = \left(\alpha - \left\lfloor \frac{k}{2} \right\rfloor k, \dots, \alpha - 2k, \alpha - k, \underbrace{\alpha, \dots, \alpha}_{\left\lfloor \frac{k+1}{2} \right\rfloor} \right), \quad \alpha = \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}.$$

Since this holds for each Θ , we have

$$J_{k,k} \leq \binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2}, \tag{17}$$

and we know what has to hold to have equality.

To complete the proof, we show by generalizing Example 3.2 that there is a Θ for which equality holds in (17).

First assume that k is even, and let D and E be the vectors in \mathbb{Z}_k^k given by

$$D = \left(\underbrace{\frac{k}{2}, \dots, \frac{k}{2}}_{\frac{k}{2}+1}, \underbrace{\frac{k}{2} + 1, \frac{k}{2} + 2, \dots, k - 1}_{\frac{k}{2}-1} \right) \quad \text{and} \quad E = \left(\underbrace{0, \dots, 0}_{\frac{k}{2}+1}, \underbrace{1, 2, \dots, \frac{k}{2} - 1}_{\frac{k}{2}-1} \right).$$

Let \widehat{D} and \widehat{E} be the matrices in \mathbb{Z}_k^k as defined in Example 3.2. Then the row sum vectors of \widehat{D} and of \widehat{E} are a permutation of one another and have the respective forms

$$U = \left(\underbrace{\binom{k}{2} + \binom{\frac{k}{2} + 1}{2}, \dots, \binom{k}{2} + \binom{\frac{k}{2} + 1}{2}}_{\frac{k}{2}}, \underbrace{*, \dots, *}_{\frac{k}{2}} \right)$$

and

$$V = \left(\underbrace{*, \dots, *}_{\frac{k}{2}}, \underbrace{\binom{k}{2} + \binom{\frac{k}{2} + 1}{2}, \dots, \binom{k}{2} + \binom{\frac{k}{2} + 1}{2}}_{\frac{k}{2}} \right).$$

Hence there exists a matrix $\Theta \in \mathbb{Z}_k^k$ with $i^*(\Theta) = \binom{k}{2} + \binom{\frac{k}{2}+1}{2}$.

Now let k be odd. We then take

$$D = \left(\underbrace{\frac{k-1}{2}, \dots, \frac{k-1}{2}}_{\frac{k+1}{2}}, \underbrace{\frac{k-1}{2} + 1, \frac{k-1}{2} + 2, \dots, k - 1}_{\frac{k-1}{2}} \right) \quad \text{and}$$

$$E = \left(\underbrace{0, \dots, 0}_{\frac{k+1}{2}}, \underbrace{1, 2, \dots, \frac{k-1}{2}}_{\frac{k-1}{2}} \right).$$

Proceeding as in the case k even, we see that there exists a matrix $\Theta \in \mathbb{Z}_k^k$ with $i^*(\Theta) = \binom{k}{2} + \binom{\frac{k+1}{2}}{2}$. \square

Corollary 3.5. *Let k and n be integers such that $k \geq 2$ and n is a multiple of k . Then*

$$J_{n,k} = \frac{n}{k} \left(\binom{k}{2} + \binom{\lceil \frac{k+1}{2} \rceil}{2} \right). \tag{18}$$

Proof. The argument in Theorem 3.4 generalizes to any multiple of k . A matrix achieving the maximum intensity is

$$J_{n/k} \otimes \Theta \quad (\text{or } \Theta \otimes J_{n/k})$$

where Θ is the matrix in the proof of Theorem 3.4 that achieves the maximum intensity when $n = k$, $J_{n/k}$ is the $(n/k) \times (n/k)$ matrix of all 1s, and \otimes denotes the tensor product. \square

We now consider the case where n is not a multiple of k . The ideas used for $n = k$ can also be used to estimate $J_{n,k}$ in general but any formula seems to be complicated and not particularly useful. For instance, a derivation like that done to obtain (12) and (13) shows that the median α satisfies

$$\frac{n}{k} \binom{k}{2} \geq \alpha - \frac{n}{k} \binom{\lceil \frac{k+1}{2} \rceil}{2}$$

hence

$$\alpha \leq \frac{n}{k} \binom{k}{2} + \frac{n}{k} \binom{\lceil \frac{k+1}{2} \rceil}{2}.$$

Taking half the entries of U very close to the median and the other half as small as possible, we obtain

$$J_{n,k} \leq \frac{n}{k} \binom{k}{2} + \frac{n}{k} \binom{\lceil \frac{k+1}{2} \rceil}{2}. \tag{19}$$

(If n is a multiple of k we have equality as given in Corollary 3.5.) For example, when $n = 6$ and $k = 12$, the grid-states with

$$D = (6, 6, 6, 6, 8, 10) \quad \text{and} \quad E = (0, 0, 0, 0, 2, 4)$$

cannot be reduced below 42 implying that $J_{6,12} \geq 42$. In general when k is even and a multiple of n , then with

$$D = \left(\underbrace{\frac{k}{2}, \dots, \frac{k}{2}}_{\frac{n}{2}+1}, \frac{k}{2} + \frac{k}{n}, \frac{k}{2} + 2\frac{k}{n}, \dots, \frac{k}{2} + \left(\frac{n}{2} - 1\right) \frac{k}{n} \right)$$

and

$$E = \left(\underbrace{0, \dots, 0}_{\frac{n}{2}+1}, \frac{k}{n}, 2\frac{k}{n}, \dots, \left(\frac{n}{2} - 1\right) \frac{k}{n} \right),$$

we get

$$J_{n,k} \geq \frac{nk}{2} + \binom{\frac{n}{2}}{2} \frac{k}{n} = \frac{5}{8}nk - \frac{k}{4}.$$

In this case where k is a multiple of n , the upper bound (19) simplifies to

$$J_{n,k} \leq \frac{5}{8}nk - \frac{n}{4},$$

and we get

$$\frac{5}{8}nk - \frac{k}{4} \leq J_{n,k} \leq \frac{5}{8}nk - \frac{n}{4}.$$

If n is bigger than k and $n = qk + r$ with $1 \leq r < k$ and r even, then using Corollary 3.5, we get

$$J_{n,k} \geq q \left(\binom{k}{2} + \binom{\frac{k}{2} + 1}{2} \right) + \frac{rk}{2} + \binom{\frac{r}{2}}{2}.$$

Similar lower bounds for $J_{n,k}$ can be obtained when n, k , and r satisfy other parity assumptions. We do not pursue this here.

4. Computation of $\mathcal{R}_{n,3}$

A formula for $\mathcal{R}_{n,2}$ is given in Theorem 3.1. An easily expressible formula for $\mathcal{R}_{n,k}$ in general does not seem possible. In this section we obtain a formula for $\mathcal{R}_{n,3}$. The method we use carries over to $\mathcal{R}_{n,k}$ but it leads to messy results.

Theorem 4.1. For $n \geq 3$, $\mathcal{R}_{n,3} = n$.

Proof. Let Θ be a grid-state in $M_n(\mathbb{Z}_3)$, and let R be the row sum vector of Θ and let S be its column sum vector. We seek the largest number w for which there is a Θ with weight w which cannot be reduced by permutation switches, that is, a coset of $M_n(\mathbb{Z}_3)$ with respect to its subgroup $P_n(\mathbb{Z}_3)$ whose weight $w^*(\Theta)$ (the smallest weight of a matrix in the coset containing Θ) is as large as possible. It follows from Corollary 2.5, that the coset $C(\Theta)$ containing Θ consists of all matrices in $M_n(\mathbb{Z}_3)$ whose row sum and column sum pairs are

$$(R, S); (R + (1, 1, \dots, 1), S + (1, 1, \dots, 1)); (R + (2, 2, \dots, 2), S + (2, 2, \dots, 2)).$$

If R and S are zero vectors, then $\Theta \in P_n(\mathbb{Z}_3)$ and so $w^*(\Theta) = 0$. If R and S are nonzero constant vectors with the same constant c , then adding $(1, 1, \dots, 1)$ or $(2, 2, \dots, 2)$ to R and S we get zero vectors and conclude again that $\Theta \in P_n(\mathbb{Z}_3)$ and $w^*(\Theta) = 0$.

If there is anywhere a common value $(0, 1, \text{ or } 2)$ in R and S , then we can put that value at the intersection of the corresponding row and column, and then 0s everywhere else in that row and column, leaving an $(n - 1) \times (n - 1)$ matrix to be specified inductively. Thus we may assume that R and S contain no common value, and thus without loss of generality that $S = (a, a, \dots, a)$ where $a \in \mathbb{Z}_3$. Suppose that R has a pair of distinct values b and c (necessarily now different from a). Then the possibilities are $a = 0, b = 1, c = 2$ or $a = 1, b = 0, c = 2$, or $a = 2, b = 0, c = 1$. Then the respective 2×2 matrices below, with two nonzero entries, at the intersection of pairs of columns and the rows corresponding to where these values are in R ,

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} (a = 0, b = 1, c = 2), \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (a = 1, b = 2, c = 0), \quad \text{and} \\ \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} (a = 2, b = 1, c = 0)$$

along with 0s everywhere in the corresponding rows and columns, leave an $(n - 2) \times (n - 2)$ matrix to be specified inductively. Thus the possibilities yet to be considered are:

- (i) $S = (0, 0, \dots, 0)$, and either $R = (1, 1, \dots, 1)$ or $R = (2, 2, \dots, 2)$,
- (ii) $S = (1, 1, \dots, 1)$, and either $R = (2, 2, \dots, 2)$ or $R = (0, 0, \dots, 0)$,
- (iii) $S = (2, 2, \dots, 2)$, and either $R = (0, 0, \dots, 0)$ or $R = (1, 1, \dots, 1)$.

Using transposition, we need only consider the first R in each of the three possibilities which we now refer to only as (i), (ii), and (iii), respectively. In each case we must have that $n = 3k$ in order that

$M_n(\mathbb{Z}_3) \neq \emptyset$. Moreover, by adding mod 3 the all 1s vector $(1, 1, \dots, 1)$ to both R and S (recall that this does not change the coset) in Case (ii) we get case (iii). Thus we only need consider Cases (i) and (iii). Taking direct sums of the following 3×3 matrices with three nonzero entries,

$$\text{(Case (i)) } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{(Case (ii)) } \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we obtain a matrix with weight n in these cases.

The above argument implies that every coset of $M_n(\mathbb{Z}_3)$ with respect to its subgroup $P_n(\mathbb{Z}_3)$ contains a matrix of weight at most equal to n . It remains to show that there is a $\Theta \in M_n(\mathbb{Z}_3)$, equivalently an R and an S for which there is a matrix $\Theta \in M_n(\mathbb{Z}_3)$ with these row and column sum vectors, such that $w^*(\Theta) = n$. Take $R = (0, 0, \dots, 0)$ and $S = (1, \dots, 1, a, b)$ where a and b are chosen in $\{1, 2\}$ so that $(n - 2) \cdot 1 + a + b \equiv 0 \pmod 3$. Then the coset consists of all matrices in $M_n(\mathbb{Z}_3)$ with row and column sum vectors

$$\begin{aligned} R &= (0, 0, \dots, 0), & S &= (1, \dots, 1, a, b) \quad \text{or} \quad R = (1, 1, \dots, 1), \\ S &= (2, \dots, 2, a +_3 1, b +_3 1) \quad \text{or}, \\ R &= (2, 2, \dots, 2), & S &= (0, \dots, 0, a +_3 2, b +_3 2). \end{aligned}$$

In all cases either R or S has no zero entries, hence the weight of the corresponding coset is at least n and so exactly n . \square

We have calculated $\mathcal{R}_{n,2}$ and $\mathcal{R}_{n,3}$. Although we do not have a general formula for $\mathcal{R}_{n,k}$, we can easily get a lower bound.

Theorem 4.2. For $n, k \geq 4$,

$$n \leq \mathcal{R}_{n,k} \leq 2n - 1.$$

Proof. There is a matrix in $M_n(\mathbb{Z}_k)$ of the form

$$\Theta = \left[\begin{array}{c|c} \begin{matrix} 1 \\ \vdots \\ 1 \\ a \\ b \end{matrix} & \begin{matrix} \\ \\ 0 \\ \\ \end{matrix} \end{array} \right],$$

where a and b are nonzero elements of \mathbb{Z}_k , and $(n - 2) \cdot 1 + a + b \equiv 0 \pmod k$ (if $n - 1 \not\equiv 0 \pmod k$ we may take $a = 1$, otherwise we choose nonzero a and b so that $a + b \equiv -(n - 2) \pmod k$). The row sum vector of $\Theta \pmod k$ is $(1, \dots, 1, a, b)$ and the column sum vector is $(0, 0, \dots, 0)$. The column sum vectors of the matrices in the coset of $M_n(\mathbb{Z}_k)$ with respect to $P_n(\mathbb{Z}_k)$ which contains Θ are $(0, 0, \dots, 0), (1, 1, \dots, 1), \dots, ((k - 1), (k - 1), \dots, (k - 1))$. Since the row sum vector of Θ is $(1, \dots, 1, a, b)$ where a and b are nonzero, it follows from Theorem 2.8 that $\mathcal{R}_{n,k} \geq n$. The upper bound is a consequence of Corollary 2.3. \square

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