Some solutions for exercises in Set # 1

1.1.25 *Petersen graph has no cycle of length 7.*

It's easy to check that there is no 3- or 4-cycle (also see book). Suppose there is a 7-cycle C. C can have no chords, otherwise we get a 3- or 4-cycle. Thus each of the 7 vertices of C is joined by an edge to the remaining 3 vertices a, b, c. One of these, say a is thus joined to three vertices of C. This easily gives a 3- or 4-cycle.

1.3.63 $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$, $\sum_{i=1}^n d_i$ even, and $d_1 \leq d_2 + \cdots + d_n$. Show there is a (multi-) graph with no loops and degree sequence $d_1, d_2, \ldots, d_n$.

If $d_1 = d_2 + \cdots + d_n$, then it's obvious what to do. Suppose that $d_1 < d_2 + \cdots + d_n$. Then since $d_1 + d_2 + \cdots + d_n$ is 0 mod 2, we must have $d_1 \leq d_2 + \cdots + d_n - 2$. Now put an edge between vertices 2 and 3, and use induction.

3. Prove that the number of spanning trees of the graph G on vertices $\{1, 2, \ldots, n\}$ obtained from the complete graph by deleting an edge is $n^{n-3}(n-2)$.

By symmetry each edge of K_n on $\{1, 2, \ldots, n\}$ is on the same number of vertices. Since a spanning tree contains $n-1$ edges, the number of pairs (T, e) with T a spanning tree and e an edge of T is, by Cayley's formula, $n^{n-2}(n-1)$. Since there are $n(n-1)/2$ edges in K_n, the number of spanning trees containing any specific edge is

$$\frac{n^{n-2}(n-1)}{n(n-1)/2} = 2n^{n-3}.$$

Hence deleting that edge we get

$$n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$$

spanning trees.