Fall Semester, 2002-03

Math 743: Exercises 3; Due Wednesday, November 13, 2002.

1. Compute explicitly the additive compound $A^{[2]} = A^{(2,1)}$ of a square matrix of order 4.

Using our formula for the entries of the additive compound we get:

$$\begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}.$$

2. If the m by n matrix A has rank r prove that the rank of its rth compound $A^{(r)}$ is 1.

By the SVD we have $A = U\Sigma V$ where U and V are unitary and the first r nonzero elements of the diagonal matrix Σ are the r nonzero singular values of A (all other diagonal elements equal 0). Then $A^{(r)} = U^{(r)}\Sigma^{(r)}V^{(r)}$. The matrix $\Sigma^{(r)}$ is a diagonal matrix with exactly one nonzero diagonal element and $U^{(r)}$ and $V^{(r)}$ are nonsingular. Hence $A^{(r)}$ has rank 1.

3. Prove that a positive semi-definite matrix A of rank r is the sum of r positive semi-definite matrices of rank 1. What is the form of a positive semi-definite matrix of rank 1?

We know that $A = U^*\Lambda U$ where Λ is the diagonal matrix with A's eigenvalues $\{\lambda_i : i = 1, 2, \ldots, n\}$ (with the nonzero eigenvalues (r of them say) coming first) on the main diagonal and U is a unitary matrix. This equation is equivalent to

$$A = \sum_{i=1}^{r} \lambda_i u_i^* u_i$$

where the u_i are the columns of U. Each rank 1 positive semidefinite is of the form in this summation.

4. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two square matrices of order n. Then their *Hadamard product* is the entry-wise product matrix $A \circ B = [a_{ij}b_{ij}]$ of order

n. Prove that if A and B are positive semi-definite (resp. positive definite) hermitian matrices then so is $A \circ B$. Hint: Look for $A \circ B$ in $A \otimes B$.

We know that $A \otimes B$ is p(s)dh if A and B are. We easily find $A \circ B$ as a principle submatrix of $A \otimes B$. Since a principal submatrix of a p(s)dh matrix is also p(s)dh, $A \circ B$ is p(s)dh.