Fall Semester, 2002-03

Math 743: Comments on solutions of Exercises 3; Due Monday, October 21, 2002.

1. Let T be a linear transformation on the space of complex matrices of order n such that T preserves the spectrum (the eigenvalues, including multiplicities). Prove that there exists a nonsingular matrix P such that $T(A) = P^{-1}AP$ for all A, or $T(A) = P^{-1}A^TP$ for all A.

Proof [This exercise was done well; in some cases there was one small error.] Since T preserves eigenvalues it must preserve determinant and so is of the form $T(A) = PAQ$ (or PA^TQ) for some nonsingular matrices P and Q. In particular, T is nonsingular (a bijection on matrices of order n). With $C = (T(I))^{-1}$ ($T(I)$ must be nonsingular since T preserves eigenvalues) we have

$$\det(\lambda I - A) = \det T(\lambda I - A) = \det(\lambda T(I) - T(A)) = \det T(I) \det(I - CA).$$

From this it follows that $\mathrm{eig}T(A) = \mathrm{eig}A = \mathrm{eig}CA$ for all A. Since T is nonsingular, every matrix of order n is of the form CX for some X, i.e. $\mathrm{eig}(X) = \mathrm{eig}(CX)$ for all X. By the polar decomposition there exist unitary U and psdh H such that $CU = H$. Now the eigenvalues of U have absolute value 1, and those of H are nonnegative reals. But by above U and H have the eigenvalues. So eigenvalues of U and H all equal 1. Since U and H are similar to diagonal matrices, they must be I and $C = I$. So $T(I) = I$ and $PQ = PIQ = I$, that is, $Q = P^{-1}$.

2. Let T be a linear transformation on the space of complex matrices of order n. Prove that T preserves the singular values (including multiplicities) if and only if there are unitary matrices P and Q such that $T(A) = PAQ$ for all A, or $T(A) = PA^TQ$ for all A..

Proof Since T preserves singular values and the rank is the number of nonzero singular values, T preserves rank and so is of the form $T(A) = PAQ$ (or PA^TQ) for some nonsingular P and Q.

Let a and b be singular values of P and Q, respectively. Let x, y be the unit vectors such that $Px = au$, and $y^*Q = bv^*$ where x, y, u, v are unit vectors. Then $A = xy^*$ has rank 1 and so has singular values $1, 0, \ldots, 0$ (since

$Ay = 1 \cdot x$), and $T(A) = Pxy^*Q$ is rank one with singular value $ab, 0, \ldots, 0$. So, $ab = 1$. From this we conclude that all singular values of P are the same and the same holds for Q. Thus, after scaling, we can assume that P and Q are unitary.

3. Let $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$. Determine

a. The singular values, left singular (real) vectors, and right singular (real) vectors of A.

b. Draw a careful picture of the unit ball in $\Re^2$ and its image under A, together with the singular vectors.

c. What are the 1-, 2-, ∞-, and Frobenius norms of A?

d. The inverse of A from the SVD.

[This was computational and caused no problem.]