MATH 475; Final Exam, 140 points, December 19, 2007 (R.A.Brualdi)

Name: These R. Solutions

Do not multiply out factorials, combination numbers, etc. Circle your answers.

I. (10+5=15 points) Let h_n be the number of n-permutation of A's, B's C's, D's, and E's where there are an even number of A's, an odd number of B's, at least one C, with no restriction on the number of D's and E's.

(1) Find a simple closed formula for the exponential generating function of $h_0, h_1, h_2, \ldots$.

$$\frac{e^x+e^{-x}}{2}\frac{e^x-e^{-x}}{2}(e^x-1)e^xe^x = \cdots = \frac{e^{5x}-e^{4x}-e^x+1}{4}$$

(2) Find a simple closed formula for h_n.

$$h_n = \begin{cases} \frac{5^n-4^n-1}{2} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

II. (30 points, 3 points each) Give something counted by each of the following numbers. Possible answers are:

1. k^n: The number of n-permutations of k objects with unlimited repetition.
2. $n!$: The number of permutations of n distinct objects.
3. $\binom{n}{k}$: The number of subsets (committees) of size k of an n element set.
4. $k\binom{n}{k}$: The number of committees of size k of an n element set with one of the elements distinguished (e.g. leader).
5. $\sum_{k=0}^{n}\binom{n}{k}^2$: The number of n-combinations of a set of $2n$ elements.
6. $n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$: The number of derangements of $1, 2, \ldots, n$.
7. C_n (Catalan numbers): The number of sequences of length 2n of $+1$'s and -1's so that each partial sum is nonnegative.
8. C_n^* (pseudo Catalan numbers): The number of ways to paranthesize multiplication of n numbers $a_1, a_2, \ldots, a_n$.
9. $s(n,k)$ (Stirling numbers of the first kind): The number of ways to partition n distinct objects into k nonempty, indistinguishable circles.
10. $S(n,k)$ (Stirling numbers of the second kind): The number of ways to partition n distinct objects into k nonempty, indistinguishable boxes.

III. (20 points) Let $\mathcal{A} = (A_1, A_2, A_3, A_4, A_5, A_6)$ be the family of subsets of $\{1, 2, 3, 4, 5, 6\}$ where

$$A_1 = \{1, 4, 5, 6\},\ \ A_2 = \{1, 4, 5, 6\},\ \ A_3 = \{1, 2, 3, 6\}$$

$$A_4 = \{1, 2, 3, 6\}, A_5 = \{1, 2, 3, 4, 5\}, A_6 = \{1, 2, 3, 4, 5, 6\}.$$

Determine the **number of SDRs**.

This is the same as the number of ways to place 6 indistinguishable nonattacking rooks on the board:

	×	×			
	×	×			
			×	×	
			×	×	
					×

.

We count this using the formula (from the I-E principal):

$$6! - r_1 5! + r_2 4! - r_3 3! + r_4 2! - r_5 1! + r_6 0!$$

where r_i is the number of ways to place i nonattacking rooks *in* the forbidden positions. These are calculated to be

$$r_1 = 9, r_2 = 28, r_3 = 36, r_4 = 20, r_5 = 4, , r_6 = 0.$$

So the answer is:

$$6! - 9 \cdot 5! + 28 \cdot 4! - 36 \cdot 3! + 20 \cdot 2! - 4 \cdot 1! + 0 \cdot 0! = 132$$

IV. (15 points) **Prove** the recurrence relation

$$S(p, k) = kS(p-1, k) + S(p-1, k-1), \quad (1 \le k \le p-1)$$

for the Stirling numbers of the second kind. (You are to use what is counted by $S(p.k)$.)

Consider $\{1, 2, \ldots, p\}$. Then $S(p, k)$ counts the number of partitions of this set into k nonempty indistinguishable boxes. These partitions are of two types: (I) p is in a box by itself. There are $S(p-1, k-1)$ of these. (II) p is not in a box by itself. Then removing p from its box, we are left with a partition of $\{1, 2, \ldots, p-1\}$ into k nonempty indistinguishable boxes. This same paritition results no matter which box contained p. Hence there are $kS(p-1, k)$ of this type. Now use the addition principle.

V. (20 points) Apply the deferred-acceptance algorithm to obtain a stable marriage in case of the preferential ranking matrix

1, 4	2, 3	3, 6	4, 2	5, 5	6, 1
3, 1	5, 2	6, 5	2, 6	1, 3	4, 4
5, 5	3, 6	6, 6	4, 4	2, 2	1, 3
6, 6	5, 5	4, 4	3, 3	2, 1	1, 2
1, 3	3, 1	5, 2	2, 5	4, 4	6, 6
4, 2	5, 4	6, 3	1, 1	2, 6	3, 4

The first entries in the positions give the preferences of the women A,B,C,D,E,F for the men a,b,c,d,e,f, while the second entries give the preferences of the men for the women. Be sure to specify the stable marriage at the end. Let the big guys do the choosing.

The algorithm gives

a to B, b to E, c to A, d to F, e to C, and f to D.

VI. (20 points) Use **Burnside's theorem** to determine the number of nonequivalent ways to color the corners of a regular 8-gon with 4 Green corners and 4 Yellow corners, under the action of the full symmetry group (D_8) of the 8-gon on its corners.

The group G contains 8 rotations (including the identity) and 8 reflections (4 each of two types). The size of $\mathcal{C}$ is $\binom{8}{4} = 70$. Counting the number of colorings fixed by the rotations we get

$$70, 0, 2, 0, , 6, 0, 2, 0.$$

Counting the number of colorings fixed by the two types of reflections we get

$$6(4 \text{ times}), 6(4 \text{ times});$$

So the numbr of nonequivalent colorings is

$$\frac{1}{16}(80 + 48) = 8.$$

VII. (20 points) A two-sided 4-omino is a 1 by 4 board with each square (8 in all because of the two sides) colored with one of the colors Red, White, Green, and Yellow. Use **Burnside's Theorem** to determine the number of different (nonequivalent) two-sided 4-ominoes.

Here there are 4 permutations in G: identity, rotate by 180 degrees, flip over, and flip over & rotate by 180 degrees. So $|G| = 4$ and $|\mathcal{C}| = 4^8 = 4296$.

The identity fixes all 4296 of the colorings. The rotation fixes $4^4 = 256$ colorings, the flip also fixes $4^4 = 256$ colorings, and the flip followed by a rotation fixes $4^4 = 64$ colorings. So the number of nonequivalent colorings is

$$\frac{1}{4}(4296 + 3 \cdot 256) = 16,576.$$